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ELEMENTS OF DYNAMICS.

THE ELEMENTS OF DYNAMICS

(*MECHANICS*)

WITH NUMEROUS EXAMPLES
AND
EXAMINATION QUESTIONS

BY

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EDINBURGH: JAMES THIN,
Publisher to the University.
1878

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PREFACE.

THE object of this treatise is to provide a manual on what has been generally known as the "Elements of Mechanics," but is here, in accordance with the more precise phraseology of recent works, termed "Dynamics."

As the work is intended for beginners, special pains have been taken to establish the necessary propositions by proofs involving no higher mathematics than the geometry of the first two books of Euclid, and algebra as far as simple equations. At the same time, the nomenclature, definitions, and general treatment are in harmony with advanced modern works on the subject.

Examples and examination questions have been introduced into the text, in order to furnish all who make use of the book with the means of testing, as they proceed, whether each portion of the subject has been duly mastered. A selection of examination papers from those set in various Universities has been added. Most of the general examples which follow each chapter are also taken from University Examinations. Answers are given in each case, and all points likely to present difficulty to beginners are explained.

PREFACE.

I gladly embrace this opportunity of thanking the many friends who have aided me. My special acknowledgments are due to Professor Balfour Stewart, of Owen's College, at whose suggestion I undertook the work; to Professor Tait, of Edinburgh University, for his ever ready encouragement and advice; and to the Rev. N. M. Ferrers, of Gonville and Caius College, and Mr J. S. Mackay, of the Edinburgh Academy, for their kind assistance in revising proof sheets.

JAMES BLAIKIE.

EDINBURGH, *February* 1878.

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CHAPTER I.

PRELIMINARY.

1. **Natural Philosophy.**—NATURAL PHILOSOPHY *treats of the Laws of the Material World.* It is concerned with the five fundamental ideas of *Time, Space, Motion, Matter,* and *Force.* Each of these ideas is the subject of a particular science. The five corresponding sciences are *Algebra, Geometry, Kinematics, Physics,* and *Dynamics.* The present work contains the elementary parts of Kinematics and Dynamics. It assumes a knowledge of Algebra as far as simple equations, and of the Geometry contained in the first two books of Euclid.

2. **Dynamics.**—DYNAMICS *treats of the action of Force.*

FORCE *is any cause which tends to alter a body's state of rest, or of uniform motion in a straight line.*

KINEMATICS *treats of Motion without reference to Force.* It is a branch of pure Mathematics, and a necessary preliminary to Dynamics.

When more forces than one act on a body, their effect is either (1) to produce or change motion, or (2) to maintain rest or prevent change of motion.

Hence Dynamics is divided into two parts—*Kinetics* and *Statics.*

KINETICS *treats of the action of Force in producing or changing Motion.*

STATICS *treats of the action of Force in maintaining Rest or preventing change of Motion.*

HYDROSTATICS *is that branch of Statics which treats of Liquids and Gases.*

[The student reading this subject for the first time is earnestly advised to make sure that he has mastered each part before he proceeds to the next. For this purpose he is urged to write out conscientiously the answers to the examinations and examples. Definitions and other sentences printed in italics should be committed to memory. If he is preparing for examination, he will find it useful to work out both examination papers and examples a second time, and to do so *with as great attention to style, accuracy, and legibility, as if he were actually undergoing examination.*]

Examination on Chapter I.

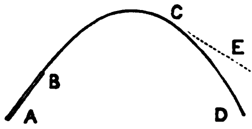
1. Enumerate the fundamental ideas of Natural Philosophy, and the respective branches of Mathematics or Natural Philosophy in which they are investigated.
2. Define Force. In what two ways is it recognised as acting?
3. Define Dynamics, Kinetics, Statics, Kinematics, and Hydrostatics.

CHAPTER II.

KINEMATICS.

3. Motion of a Point.—MOTION *is change of Position.* In Kinematics we consider the motion of a Point. A moving point passes from one Position in *Space* to another during the lapse of an Interval of *Time.* Motion therefore connects the ideas of Space and Time.

A moving point traces out a continuous Line, any small part of which is either *straight* or *curved.* If the line is straight, then its *Direction* is that in which the point is moving. If the line is curved, then we can draw a straight line which will have the direction of the motion of the point at any particular instant. This line is called in Geometry the *Tangent* to the curve.



As an example, take the case of a projectile impelled along a straight tube AB. As long as it is within the tube, its direction is that of the straight line AB. When it leaves the tube it describes the curve BCD. When the projectile is at C, its direction is that of the straight line CE, which is the tangent to the curve at C.

4. Uniform Velocity.—VELOCITY *is Rate of Motion.* It is *Uniform* or *Variable.*

UNIFORM VELOCITY *is the Rate of Motion of a point which* *passes over equal spaces in equal times.* It is greater or less as the space passed over in a given time is greater or less.

Thus a ship which, sailing uniformly, makes 15 knots in a certain time, has three times the velocity of one which makes 5 knots in the same time.

Uniform Velocity is measured by the space passed over in a unit of time. It is convenient to take a *foot* as the unit of space, and a *second* as the unit of time. A point has therefore unit velocity, when it moves one foot in each second, a velocity of 12 when it moves 12 feet in each second, and so on.

Example.—Find the velocity of a train moving uniformly at the rate of twenty miles an hour.

We must first express the distance in feet, and the time in seconds. Employing factors,

$$20 \text{ miles} = 20 \times 1760 \times 3 \text{ feet.}$$
$$1 \text{ hour} = 60 \times 60 \text{ seconds.}$$

The data may therefore be thus stated—

In 60 × 60 sec. the train moves 20 × 1760 × 3 feet.

$$\therefore \quad ,, \quad 1 \quad ,, \quad ,, \quad \frac{20 \times 1760 \times 3}{60 \times 60} \quad ,,$$

∴ The velocity, or the space passed over in one second, $=\frac{20 \times 1760 \times 3}{60 \times 60}$.

This may be written $\frac{20 \times 88 \times 20 \times 3}{20 \times 3 \times 20 \times 3}$, which, on cancelling factors common to numerator and denominator, becomes

$$\frac{88}{3} = 29\tfrac{1}{3}, \text{ the velocity required.}$$

Examples for Exercise.—

1. Find the velocity of a horse which trots 2 miles in 9 minutes.

2. Which is the greater velocity, 100 yards in $11\frac{1}{3}$ seconds, or $6\frac{3}{4}$ miles in 22 minutes?

3. The velocity of sound is 1118. How long does it take to travel 43 miles?

5. Algebraical Formulæ of Uniform Velocity.—Let v be the velocity of a point moving uniformly, that is, the number of feet which it passes over in each second. Then

In 2 seconds it passes over $2v$ feet.
" 3 " " " $3v$ "
" 4 " " " $4v$ "

and so on.

Let t be any number of seconds. In t seconds the point passes over $t \times v$ feet, which may be written tv or vt. If this number of feet be called s, then

$$vt = s, \text{ or } s = vt \quad . \quad . \quad . \quad . \quad (1)$$

This equation enables us to find s when v and t are known.

Divide both sides of equation (1) by t and we obtain the equation

$$v = \frac{s}{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which enables us to find v when s and t are known.

Again, divide both sides of (1) by v and we obtain

$$t = \frac{s}{v} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which enables us to find t when s and v are known.

Examples—

1. The velocity of a particle is 11. Find the distance gone in 7 seconds.

From (1) $s = vt$
but $v = 11, \ t = 7$
$\therefore s = 11 \times 7 = 77$ feet.

2. A train goes a mile in a minute. Find its velocity.

From (2) $v = \dfrac{s}{t}$

but $s = 1760 \times 3, \quad t = 60$

$$\therefore v = \frac{1760 \times 3}{60} = 88.$$

3. A race horse has a velocity of 45. Find how long he takes to go half a mile.

From (3) $t = \dfrac{s}{v}$

but $s = 880 \times 3, \quad v = 45.$

$$\therefore t = \frac{880 \times 3}{45} = 58\tfrac{2}{3} \text{ seconds.}$$

Examples for Exercise.—

1. Find the distance an eagle flies in an hour, assuming its velocity to be 100.

2. Find the velocity of the moon in its orbit round the earth, assuming that it describes a path of 1,511,460 miles in 27 days.

3. Find the time in which a body moving with uniform velocity a passes over b feet.

6. Variable Velocity.—VARIABLE VELOCITY *is that of a point which passes over unequal spaces in equal times. It is measured at any instant by the space which would be passed over in a unit of time, were the velocity at the instant under consideration to remain unchanged for a unit of time.* Thus when we say that a bullet leaves a rifle with a velocity of 1330, we mean that, if the velocity were to remain unaltered for a second, the bullet would, during that time, describe a path 1330 feet in length.

7. Average Velocity.—*When a point moves with variable velocity for any interval of time, the Uniform Velocity, with* *which the same space can be described in the same time, is called the* AVERAGE VELOCITY *of the point during that time.*

During part of the interval the point's velocity must have been greater than the average velocity, during part it must have been less, and at some one instant it must have been equal to it. The shorter the interval is, the more nearly does the average velocity coincide with the actual velocity at any instant of the interval.

A railway train, either increasing or decreasing its speed, is an example of variable velocity. By observing how far it goes in five minutes, we may obtain the average velocity during that period. A nearer approximation to its actual velocity at a particular instant is found by observing the distance travelled in one minute, chosen so as to include that instant, and a still nearer by observing that travelled in one second similarly chosen.

Examples.—

1. A man walks 9 feet in one second, 8 in the next, 6 in the third, and 3 in the fourth; find his average velocity.

Here we have to find a uniform velocity with which the whole distance, 26 feet, can be described in 4 seconds.

By (2) $v = \frac{s}{t} = \frac{26}{4} = 6\frac{1}{2}$, the average velocity required.

2. Find approximately the same man's velocity at the end of the first second, assuming that he is constantly slackening his pace.

The smallest interval, which we can take so as to include the given instant, is the first two seconds. During this time he walked 17 feet.

$\therefore v = \frac{s}{t} = \frac{17}{2} = 8\frac{1}{2}$, the average velocity during the first two seconds.

Examples for Exercise.—

1. A stone falls 16·1 feet in the first second, 48·3 in the next, and 80·5 in the third; find its average velocity.

2. With the same data, find as nearly as possible the stone's velocity at the beginning of the third second.

3. A particle passes over a feet in the first second, b in the next, and c in the third. Find its average velocity.

4. A particle has during t_1 seconds an average velocity v_1, and during the next t_2 seconds its average velocity is v_2. Find its average velocity during the whole time.

8. Representation of Velocities by Straight Lines.—The motion of a point at any instant is defined when its *direction* and *rate* are given. It is therefore completely represented by a Straight Line whose *direction* is that in which the particle is moving, and whose *length* is proportional to its velocity at the instant under consideration.

Examination on Sections 3—8.

1. Define Motion. Show how it involves space and time.

2. Define Velocity. What two kinds of velocity exist? Give examples of each.

3. Define Uniform Velocity. How is it measured?

4. Prove the formula $s = vt$. Deduce from this a formula which expresses the velocity in terms of the space and time.

5. Define Variable Velocity. How is it measured?

6. What is meant by the Average Velocity of a point during a given time?

7. Show that a Velocity may be completely represented by a straight line.

9. Velocity of a Body.—When a material body, or a solid figure, is in motion, if all its points have the same velocity at any instant, that is if they are all moving at the same rate in parallel straight lines, that velocity is called *the velocity of the body.* A canal boat moving along a straight canal, and the framework of a carriage drawn along a straight and level road, are examples of bodies in which all the points have the same velocity. A revolving wheel and a rolling ball are examples of bodies in which the different points have different velocities, but we shall not require to consider the motion of such bodies.

10. Component and Resultant Velocities.—*When a point has two or more simultaneous velocities, each of them is called a Component, and the actual velocity is called their Resultant.*

The velocity of a point is always found with regard to some body which we suppose to be at rest. When a sailor climbs the rigging of a ship, we may find his velocity with regard to the ship. If we combine with this velocity the velocity of the ship, we shall find the sailor's velocity with regard to the surface of the earth.

11. Composition of two uniform Velocities in the Straight Line.—Two velocities in the same straight line may have the same direction, or they may have opposite directions. There are therefore two cases to be considered.

(1) *If two component velocities have the same direction in the same straight line, the Resultant is their Sum.*

Let a point have two uniform velocities, v_1 and v_2, in the same straight line. Then since v_1 and v_2 are uniform, or the same at every instant, the resultant velocity is also the same at every instant, that is to say, it is a uniform velocity.

To find its magnitude let us suppose that the point moves with the velocity v_1 along the straight line AB, and that the page is itself in motion in the same direction with the velocity v_2. (See § 9.) After one second the point will have moved v_1 feet along AB, but AB will itself have moved v_2 feet in the same direction owing to the motion of the

page. The point will therefore now be $v_1 + v_2$ feet from its original position in space, or if v be its actual velocity

$$v = v_1 + v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(2) *If two component velocities have opposite directions in the same straight line the Resultant is their Difference, and has the direction of the greater component.*

Let a point have two uniform velocities, v_1 and v_2, in opposite directions. Then, as before, the resultant velocity is uniform. To find its magnitude, let us again suppose that the point is moving along AB with velocity v_1, but that the page is moving in the direction of BA, that is, in the opposite direction, with velocity v_2. If v_1 be greater than v_2, the point will, after one second, be $v_1 - v_2$ feet from its original position in the direction of AB, or, if v be its actual velocity,

$$v = v_1 - v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If v_2 be greater than v_1, the point will, after one second, be $v_2 - v_1$ feet from its original position in the direction of BA. It is not necessary to form a new equation in this case, for equation (5) may be put into the form

$$v = -(v_2 - v_1),$$

which will apply to this case if we agree to consider that a negative velocity means velocity in the opposite direction to that chosen as positive.

Corollary.—If a point have two equal and opposite velocities in the same straight line, it is at rest.

Proof.—For if in (5) $v_1 = v_2$,
then $v = v_1 - v_2 = 0$,

or the point is at rest.

Examples.—

1. A ship is sailing with velocity 15, and a sailor walks along the deck in the direction of the ship's motion with velocity 6. Find the actual velocity of the sailor.

Here $v_1 = 15,\ v_2 = 6,$
∴ By (4) $v = v_1 + v_2 = 21.$

2. Find the sailor's actual velocity when he walks at the same rate in the opposite direction.

Here $v_1 = 15$ and $v_2 = 6$ in the opposite direction,
∴ By (5) $v = v_1 - v_2 = 9.$

3. Find the sailor's actual velocity when he runs with velocity 32 in the direction opposite to the ship's motion.

Here $v_1 = 15,\ v_2 = 32$ in the opposite direction,
∴ By (5) $v = v_1 - v_2 = 15 - 32 = -17,$

or the real velocity of the sailor is 17 in the direction opposite to the ship's motion.

Examples for Exercise.—

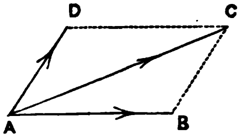
1. A stream has a velocity of 2 miles an hour, and a boat is rowed first up stream and then down at the rate of 8 miles an hour. Find its actual velocity in each case.

2. A sledge party travels northwards on an ice-floe at the rate of 12 miles a day. The floe is itself drifting southwards at the rate of 15 yards a minute. In what direction is the sledge really moving, and at what rate?

3. A river steamer sails down stream at the rate of 15 miles an hour, and up stream at the rate of 11 miles an hour. Find the velocity of the river, and that of the steamer relatively to it.

12. Parallelogram of Velocities.—*If two component velocities be represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant is represented by the diagonal passing through their intersection.*

Let a point have two *uniform* velocities represented by the straight lines AB and AD. Complete the parallelogram ABCD. Join AC. AC represents the resultant velocity.



Proof. — Since the component velocities are uniform, or the same at each instant, their resultant is also the same at each instant both in magnitude and direction, that is to say, it is a uniform velocity in a straight line.

To find it, let us suppose that the point moves with velocity AB along the straight line AB, and that the page is in motion in the direction AD with velocity AD. After one second the point will have moved along AB from A to B, but the line AB will itself have moved into the position DC, owing to the motion of the page, and therefore the point will now be at C. It has therefore moved in the direction AC with velocity AC.

13. Triangle of Velocities.—*If a point have three component velocities, which are represented in magnitude and direction by the sides of a triangle taken in order, the point is at rest.*

Let a point have three uniform velocities represented by the sides AB, BC, CA of the triangle ABC.

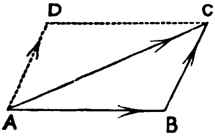


Figure 1.

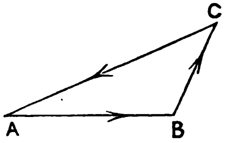


Figure 2.

In the parallelogram ABCD, BC is equal and parallel to AD, and thus equally represents the component velocity in that direction. Therefore in the triangle ABC (figure 1), the velocity AC is the resultant of the two velocities AB and

BC. If we combine with these the velocity CA (figure 2), then the two velocities AC and CA are equivalent to the three velocities AB, BC, and CA. But AC and CA are two equal and opposite velocities. Therefore the point is at rest.

14. Polygon of Velocities.—*If several component velocities be represented by all but one of the sides of a polygon taken in order, their resultant is that side taken in the opposite direction.*

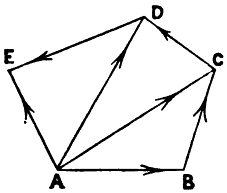


Figure 1.

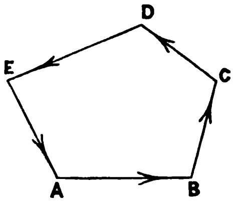


Figure 2.

Let AB, BC, CD, DE (figure 1) be the component velocities. Then AC is the resultant of AB and BC.

AD is the resultant of AC and CD, that is, of AB, BC, and CD.

AE is the resultant of AD and DE, that is, of AB, BC, CD, and DE.

Corollary.—If a point have component velocities represented by all the sides of a polygon taken in order it is at rest.

Let a point have velocities represented by AB, BC, CD, DE, EA.

AE is the resultant of AB, BC, CD, and DE.

∴ AE and EA are equivalent to AB, BC, CD, DE, and EA.

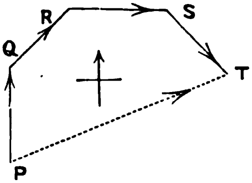
But AE and EA are two equal and opposite velocities.

∴ The point is at rest.

NOTE.—The above proofs hold good when the velocities, and therefore the sides of the polygon, are not all in one plane.

Example.—

A point has the following simultaneous velocities, 8 north, 7 north-east, 8 east, and 7 south-east. Find the resultant velocity.



Draw a line PQ north = 8
QR north-east = 7
RS east = 8
ST south-east = 7

PT is the resultant velocity. It will be found by measurement to be rather greater than 19.

Examples for Exercise.—

1. Find by means of a figure the resultant of the following velocities, 6 north, 2 east, 2 south, 5 west.

2. Find the resultant of velocities 3, 4, and 5 in the directions of the sides of an equilateral triangle taken in order.

Note.—The advanced student may obtain the same results by calculation.

Examination on Sections 9—14.

1. When can a material body be said to have a certain velocity? Give an example of a body which moves so that its different parts have different velocities.

2. Define component and resultant velocities, and give examples.

3. Prove that, if two component velocities are in the same direction, the resultant is their sum.

4. Prove that, if two component velocities are in opposite directions, the resultant is their difference.

5. Prove that a point which has two equal and opposite velocities is really at rest.

6. Enunciate the proposition known as the Parallelogram of Velocities.

7. Prove that, if two velocities be represented by two adjacent sides of a parallelogram, the resultant is the diagonal through their point of intersection.

8. Enunciate the proposition known as the triangle of velocities.

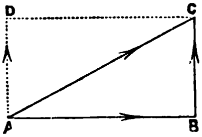
9. Prove that if two velocities are represented by two sides of a triangle taken in order, the third side of the triangle is their resultant.

10. Prove that the resultant of three velocities, which are proportional and parallel to the sides of a triangle taken in order, is zero.

11. Show how to find the resultant of any number of velocities co-existing in a point.

12. When any number of simultaneous velocities exist in a point, what is the condition that their resultant vanishes?

15. Resultant of two Velocities at Right Angles.—If the component velocities AB and AD, or AB and BC are at right angles, the parallelogram ABCD is a rectangle, and the triangle ABC is right angled.



By Geometry (Euclid I., 47).

$$AC^2 = AB^2 + BC^2$$

or, if v_1 and v_2 be the components and v the resultant velocity,

$$v^2 = v_1^2 + v_2^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

hence, extracting the square root of each side,

$$v = \sqrt{v_1^2 + v_2^2}$$

Example.—

A ship sails eastwards with velocity 4, and is at the same time carried northwards by a current with velocity 3. Find its real velocity.

Here $v_1 = 4 \;\therefore\; v_1^2 = 16$

$v_2 = 3 \;\therefore\; v_2^2 = 9$

And by (6) $v^2 = v_1^2 + v_2^2 = 25 \;\therefore\; v = 5.$

Also the direction of the ship's real motion is that of AC in the above figure if AB = 4 and BC = 3.

Examples for Exercise.—

1. A ship is sailing at the rate of 12 miles an hour. A sailor

walks across the deck at the rate of 5 miles an hour. Find his resultant velocity, showing its direction by means of a figure.

2. A man swims across a river with velocity 3·6. The velocity of the current is 10·5. Find the real velocity of the man.

3. If the river in the last example is 972 feet broad, find how far the man is carried down by the current.

16. Resultant of two Equal Velocities.—*If two component velocities be equal, the resultant will bisect the angle between them.*

Proof.—In the parallelogram of velocities ABCD, let AB = AD, then BC also = CD, and the triangles ABC, ADC have three sides equal.

∴ the angle DAC = the angle BAC (Euclid I., 8).

17. Resultant of any two Velocities.—Let AB and AD, or AB and BC, be any two velocities whose directions form a given angle DAB, and let AC be their resultant.

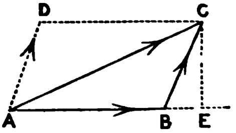


Figure 1.

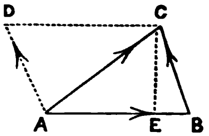


Figure 2.

From C draw CE perpendicular to AB, meeting it, produced if necessary, in E. By Geometry

(1) If DAB be acute, as in figure 1, then ABC is obtuse.

$\therefore\ AC^2 = AB^2 + BC^2 + 2AB \cdot BE$ (Euclid II., 12), . (7)

(2) If DAB be obtuse, as in figure 2, then ABC is acute.

$\therefore\ AC^2 = AB^2 + BC^2 - 2AB \cdot BE$ (Euclid II., 13), . (8)

Equation (7), or (8), enables us to calculate the value of AC, provided we know the length of BE. This can be found by elementary Geometry when the angle DAB is one of the following :—30°, 60°, 120°, 150°, 45°, 135°.

In the first four cases the triangle BCE is the half of an equilateral triangle, and its hypotenuse BC is equal to twice its shortest side; in the last two cases CE is equal to BE; thus in all cases BE can be found. We give an example of each kind of triangle. The second example, in this and in the following section, may be omitted by those who do not understand surds.

Examples.—

1. Find the resultant of two velocities 10 and 6 whose directions form an angle of 60°.

In figure 1, let $AB = 10$, $BC = 6$, $\angle DAB = \angle CBE = 60°$. Then CBE is the half of an equilateral triangle.

$$\therefore BE = \tfrac{1}{2} BC = 3.$$

By (7) $$AC^2 = AB^2 + BC^2 + 2AB \cdot BE.$$
$$= 100 + 36 + 60 = 196 \therefore AC = 14.$$

2. Find the resultant of velocities 7 and 4, making an angle of 135°.

In figure 2, let $AB = 7$, $BC = 4$, $\angle DAB = 135°$.

Then $$\angle CBA = 180° - 135° = 45°.$$
$$\therefore \angle BCE = 45°.$$
$$\therefore CE = BE.$$

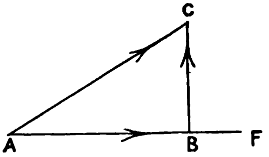
and $$BC^2 = BE^2 + CE^2 = 2BE^2$$
$$\therefore BE^2 = \frac{1}{2} BC^2$$
$$\therefore BE = \frac{1}{\sqrt{2}} BC = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

By (8) $$AC^2 = AB^2 + BC^2 - 2AB \cdot BE$$
$$= 49 + 16 - 28\sqrt{2} = 65 - 28\sqrt{2}$$
$$\therefore AC = \sqrt{65 - 28\sqrt{2}}.$$

Examples for Exercise.—

1. Find the resultant of velocities 7 and 8 at an angle of 60°.
2. Find the resultant of velocities 8 and 5 at an angle of 120°.
3. Find the resultant of two unit velocities—(1) at 30°, (2) at 45°, (3) at 150°.

18. Resolution of a Velocity in two Directions at Right Angles.—Let AC be any velocity. Through A draw any straight line AF, and from C draw CB perpendicular to AF. Then the velocity AC is the resultant of the velocities AB and BC. These velocities are called the resolved parts of AC along and perpendicular to AF.



Examples.—

1. Let AC be a velocity of 5, and let AB, one of its rectangular components, be equal to 4. Find BC.

$$BC^2 = AC^2 - AB^2 = 25 - 16 = 9$$
$$\therefore BC = 2$$

2. Let AC be a unit velocity, and let the $\angle$ CAF = 30°. Find the components of AC along and perpendicular to AF.

The angles CAB and ACB together make up a right angle.

$$\therefore \angle ACB = 60^\circ$$

$\therefore$ ACB is half an equilateral triangle, and $CB = \frac{1}{2} AC = \frac{1}{2}$.

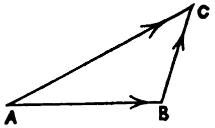
also $$AB^2 = AC^2 - CB^2 = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$
$$\therefore AB = \tfrac{1}{2}\sqrt{3}.$$

Examples for Exercise.—

1. A vertical velocity is equal to 4. Find its component along a line which makes an angle of 30° with the horizon.

2. A velocity of $3\sqrt{2}$ is in a direction inclined at an angle of 45° to the horizon. Find its horizontal component.

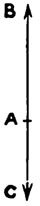
19. Composition and Resolution of Variable Velocities. Variable velocities may be compounded and resolved in the same manner as has been shown to hold good for uniform velocities. For the effect of a variable velocity at any instant is the same as that of an equal uniform velocity, and in compounding we are only concerned with the instantaneous effect.



20. Change of Velocity.—Let the velocity of a point at given instant be AB. If the velocity be uniform AB will continue to represent it, but if it be variable then after an interval of time, it will be found to have changed in magnitude, or in direction, or in both.

Let AC be the new velocity. The change in the velocity is the same as would have been caused by combining with AB a velocity BC. BC is therefore called the change of velocity during the interval. The change of velocity is defined when we know its *direction* and *magnitude.*

Examples.—



1. A stone is thrown up with velocity 12. After an interval it is found to be descending with velocity 8. Find its change of velocity.

From A draw AB to represent the original, AC the changed velocity. BC will represent the change of velocity.

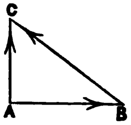
Here AB is vertically upwards $= 12$
AC „ „ downwards $= 8$
$\therefore$ BC „ „ „ $= 20$.

2. A velocity of 4 east is changed into 3 north. Find the change of velocity.

Drawing the figure we see that the triangle ABC is right angled at A.

$$\therefore BC^2 = AB^2 + AC^2 = 16 + 9 = 25$$
$$\therefore BC = 5.$$

Its direction is given by the figure.



Examples for Exercise.—

1. A point revolves in a circle with unit velocity. Find the change in velocity during an interval in which an arc of 60° is described.

2. Find the change in velocity during an interval in which the same point describes a quadrant.

21. Acceleration.—*When the velocity of a point varies, the rate of change is called the* ACCELERATION. Like Velocity it is *uniform* or *variable.* UNIFORM ACCELERATION *is measured by the change of velocity in a unit of time.* A point has therefore unit acceleration when it receives in each second a change of velocity (see § 20) of one foot per second. Like velocities, accelerations are represented by straight lines in their directions, and proportional to their magnitude. They are compounded and resolved in the same way as velocities. A velocity is said to be *accelerated* whenever it is changed either in magnitude or in direction, or in both; thus a velocity undergoing acceleration may have its magnitude increased or diminished, or its direction may be changed while its magnitude is left unaltered.

22. Uniform Acceleration in the Direction of Motion.—Let a point be in motion with velocity V, and let it have a uniform acceleration α in the direction of motion. Then the increase of velocity in each second is equal to α.

∴ At the beginning its velocity is V.
After one second „ is $V + \alpha$.
„ two seconds „ is $V + 2\alpha$.
„ three „ „ is $V + 3\alpha$, and so on.
„ t „ „ is $V + t\alpha$ or $V + \alpha t$.

If, then, the velocity after t seconds be called v,

$$v = V + \alpha t \quad . \quad . \quad . \quad . \quad . \quad (9)$$

To find the space passed over, we must first find the average velocity. Since the velocity increases uniformly, the average velocity during any interval is equal to the velocity at the middle of that interval; or, what is the same thing, half the sum of the velocities at the beginning and end of the interval.*

---

* The same is true of any quantity which increases uniformly. Thus, the average value of a man's age between his 20th and 30th year is $\frac{20 + 30}{2}$ or 25, and is so solely because age is a quantity which increases uniformly.

Thus, to find the average velocity during t seconds, we have—

$$\begin{aligned} &\text{Velocity at beginning} &&= V. \\ &\text{Velocity at end} &&= V + at. \\ \hline &\therefore \quad \text{Their sum} &&= 2V + at. \end{aligned}$$

Dividing this sum by two, we find the average velocity

$$= V + \tfrac{1}{2}\,at.$$

Also, if s be the space passed over, then

$$\begin{aligned} s &= (V + \tfrac{1}{2}\,at)\,t \\ &= Vt + \tfrac{1}{2}\,at^2 \quad . \quad . \quad . \quad . \quad . \qquad (10) \end{aligned}$$

Examples.—

1. A body has a uniform acceleration 12. If its initial velocity be 7, find its velocity after 5 seconds.

By (9) $\qquad v = V + at$

here $\qquad V = 7,\ a = 12,\ t = 5.$

$\therefore v = 7 + 12 \times 5 = 67.$

2. Find the space which the body passes over during the five seconds.

By (10) $\qquad s = Vt + \frac{1}{2}\,at^2$

$= 7 \times 5 + \frac{1}{2} \times 12 \times 25 = 35 + 150 = 185$ feet.

3. A body starts from rest with a uniform acceleration g. Find its velocity after t seconds, and the space passed over in that time.

Here $\qquad a = g$, and $V = 0$

By (9) $\qquad v = V + at = 0 + gt$

By (10) $\qquad s = Vt + \frac{1}{2}\,at^2 = 0 + \frac{1}{2}\,gt^2.$

Hence we obtain the equations $v = gt$, $s = \frac{1}{2}\,gt^2$.

We shall afterwards find that this is the case of a body falling under the action of gravity.

4. A body starts from rest with a uniform acceleration of 32·2. Find its velocity after 8 seconds.

Here the equations are these of example (3).

$$v = gt$$

$$s = \tfrac{1}{2}gt^2$$

Also $g = 32{\cdot}2, \quad t = 8$

$$\therefore v = 257{\cdot}6 \quad \text{and} \quad s = 1030{\cdot}4.$$

Examples for Exercise.—

1. A body starts with velocity 7, and has a uniform acceleration $\frac{1}{3}$. Find the velocity after 12 seconds, and the space passed over in that time.

2. A body starts from rest, and has a uniform acceleration 10. Find in what time it describes 125 feet, and what is its velocity at the end of that time.

3. A body starts with a velocity 48·3, and has a uniform acceleration 32·2. How far will it travel in 10 seconds, and what will be its velocity after that interval?

23. Uniform Acceleration opposite to the Direction of Motion.—Let a point be in motion with velocity V, and let it have a uniform acceleration α opposite to the direction of motion. Then the decrease of velocity in each second is α. Therefore the velocity after t seconds is $V - \alpha t$, or

$$v = V - \alpha t \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Also, the average velocity $= V - \frac{1}{2}\alpha t$, and therefore the space passed over in t seconds is $(V - \frac{1}{2}\alpha t)\,t$, or

$$s = Vt - \tfrac{1}{2}\alpha t^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Examples—

1. A body has a velocity 10, and an acceleration 2 opposite to its motion. Find how far it will go in 3 seconds, and its velocity at the end of that time.

Here by (11) $v = V - at$

$$= 10 - 2 \times 3 = 4.$$

Also by (12) $s = Vt - \frac{1}{2}at^2$

$$= 10 \times 3 - \tfrac{1}{2} \times 2 \times 9 = 21.$$

2. Find after what time the body will come to rest.

Here we have to find what value of t makes $v = 0$.

That is $V - at = 0$.

This may be written $V = at$,

Or $at = V$

$$\therefore t = \frac{V}{a} = \frac{10}{2} = 5 \text{ seconds.}$$

3. A body starts with velocity V in a certain direction, and has a uniform acceleration g in the opposite direction. Investigate its motion.

Here (11) and (12) take the forms.

$$v = V - gt$$
$$s = Vt - \tfrac{1}{2}gt^2.$$

From these equations we see that v, which is at first $= V$, gradually gets smaller until it is equal to nothing, when $t = \frac{V}{g}$. After that v is negative, or the body begins to move in the direction opposite to its former motion. If we substitute for t in the second equation the value $\frac{V}{g}$, we obtain for s the value $\frac{V^2}{2g}$, which is the farthest point reached by the body.

We shall afterwards find that this is the case of a body thrown upwards, and acted on by gravity.

4. A body starts with velocity 80·5, and a uniform acceleration 32.2 opposite to its direction of motion. Find its velocity and position after 4 seconds. Here the above equations give

$$v = 80{\cdot}5 - 32{\cdot}2\times4$$
$$= -48{\cdot}3$$
$$s = 80{\cdot}5\times4 - \tfrac{1}{2}\times32{\cdot}2\times16$$
$$= 64{\cdot}4.$$

∴ After 4 seconds the body will be returning with a velocity 48·3, and will be 64·4 feet from the starting point.

Examples for Exercise.—

1. A body has a velocity 16, and a uniform acceleration 3 opposite to the direction of motion.
   (1) How far will it go in 3 seconds?
   (2) What will be its velocity after 3 seconds?

(3) When will the body stop?

(4) How far will it go before stopping?

(5) What will be its velocity and distance after 8 seconds?

(6) When will it return to its starting point, and with what velocity?

2. A body starts with velocity ng, and has a uniform acceleration g opposite to the direction of motion.

(1) Show that it will be at rest after n seconds.

(2) Find the distance reached in that time.

(3) Show that it will take n seconds to return from that distance.

(4) Show that it will reach the starting point again with the same velocity with which it left it.

Examination on Sections 15—23.

1. Find the resultant of two velocities at right angles.

2. Prove that the resultant of two equal velocities bisects the angle between them.

3. Find the resultant of any two velocities whose directions form an acute angle.

4. Find the resultant of any two velocities whose directions form an obtuse angle.

5. What is meant by resolving a velocity in two directions at right angles?

6. Prove that variable velocities may be compounded and resolved in the same manner as uniform velocities.

7. Define change of velocity.

8. Define acceleration. How is it measured when uniform?

9. Prove that with uniform acceleration a in the direction of motion, if V be the initial velocity and v the velocity after t seconds,

$$v = V + at.$$

10. When a point moves with uniformly accelerated velocity during any interval of time, how is the average velocity obtained? At what time was the point moving with the average velocity?

11. Find the space passed over in time t by a point moving with initial velocity V and uniform acceleration in the direction of motion.

12. Form the equations of motion of a point which has uniform acceleration opposite to the direction of motion.

General Examples in Kinematics.

[It is not necessary that the student should solve all the general examples. He should merely write out a sufficient number to make sure that he has mastered the principles of the chapter. The others may be useful when he comes to revise.]

1. Express the following velocities in feet per second.

(1) 12 miles an hour.
(2) 2500 miles in 24 hours.
(3) 9 inches in 14 minutes.
(4) 25 feet a minute.
(5) 100 miles in an hour and a half.
(6) 5 yards a second.
(7) 10 feet in $\frac{1}{5}$ of a second.
(8) 75 feet in 2 hours.
(9) 91,760,000 miles in 8 minutes and 16 seconds.
(10) $\frac{1}{2}$ of $\frac{1}{3}$ of an inch in $\frac{1}{8}$ of $\frac{1}{9}$ of a second.

2. The following velocities are in feet per second. Express them in miles per hour.

(1) Channel steamer 22.
(2) Carrier pigeon 120.
(3) Sound 1122. (This varies with the temperature, &c.)
(4) A rifle bullet 1342.
(5) The earth's centre 101,090.

3. Find the number of miles travelled in 20 minutes by bodies moving with the velocities mentioned in question 2. (5 answers).

4. Find the number of feet travelled in $\frac{1}{10}$ of a second by the same bodies. (5 answers).

5. Find the number of seconds each of these bodies takes to pass over 22 feet. (5 answers).

6. Find the velocity of a point which passes over—

(1) a feet in b seconds.
(2) $2a$ feet in $3a$ seconds.
(3) a yards in b minutes.
(4) 10 feet in $a + b$ seconds.
(5) a^2b feet in ab^2 seconds.
(6) $a^2 - b^2$ feet in $a + b$ seconds.

7. Find how many seconds a point takes to describe a circle with uniform velocity in the following cases:—

(1) Radius 1, velocity 1.
(2) „ a, „ v.
(3) Radius $\frac{1}{2}$, velocity π.
(4) „ 7, „ 3π.

Note.—π represents the ratio of the circumference to the diameter of a circle, or $2\pi r$ is the circumference of a circle whose radius is r.

8. A man walks 3 miles in an hour, and a fourth mile in 12 minutes. Find his average velocity—

(1) in feet per second.
(2) in miles per hour.

9. A man walks 10 miles in $2\frac{1}{2}$ hours, walking for the first $\frac{3}{4}$ of an hour at the rate of 5 miles an hour. Find his average velocity during the remainder of the time—

(1) in feet per second.
(2) in miles per hour.

10. If the velocity of a bullet is 1250 and that of sound 1200, how much time will elapse, on a range of 1000 yards, between the time that the bullet strikes the target and the time that the sound of the discharge reaches the target?

11. A walks along a road at the rate of 4 miles an hour. After he has gone 7 miles B follows him, walking at the rate of 5 miles an hour. How far will B have walked when he overtakes A?

12. Compound the following velocities in the same straight line:—

(1) 3, 5, 7, and 12, all forwards.
(2) 6, 8, and 9, all backwards.
(3) 4 and 7 forwards, 9 backwards.
(4) 7 forwards and 13 backwards.
(5) $a + b$ forwards and $a - b$ backwards.
(6) $x(x-1)(x-2)$ forwards and $x(x+1)(x+2)$ backwards.

13. A boat is rowed at the rate of 6 miles an hour in still water. How long will it take to row it 5 miles up a river against a current flowing 5 feet in a second?

14. If the boat (in last example) be rowed down the same stream, how far will it go in an hour?

15. A runs 100 yards in 12 seconds, B runs a mile in 5 minutes. Find the ratio of their velocities.

16. Find the resultant of two velocities, v_1 and v_2, whose directions are at right angles when their values are—

(1) 6 and 8.
(2) 1, $\frac{3}{4}$.
(3) 8, 15.

(4) $5x$, $\frac{11}{12}x$.
(5) 20, 21.
(6) $$\left.\begin{aligned} v_1 + v_2 &= 17 \\ v_1 - v_2 &= 7 \end{aligned}\right\}$$
(7) $$\left.\begin{aligned} 2av_1 + bv_2 &= a^3 \\ av_2 - 2bv_1 &= b^3 \end{aligned}\right\}$$

17. Find by means of accurately drawn figures the directions and magnitudes of the resultants of the following velocities:—
    (1) 3 and 4 at 45°.
    (2) 4 and 6 at 105°.
    (3) 2, 5, and 7 in the directions of the sides of an equilateral triangle taken in order.
    (4) 3, 4, and 5 in the directions of three consecutive sides of a square.
    (5) 2, 3, 2, 4, and 3 in the directions of 5 consecutive sides of a regular octagon.

18. Find the resultants of the following pairs of velocities at an angle of 60°:—
    (1) 5 and 16.
    (2) 1, $1\frac{1}{7}$.
    (3) $2x + 1$, $x(3x + 2)$.

19. Find the resultants of the following pairs of velocities at an angle of 120°:—
    (1) 16 and 21.
    (2) $8\frac{1}{4}$, 10.
    (3) $2x + 1$, $x(x + 2)$.

20. Find the resultants of velocities in the directions of the sides of an equilateral triangle taken in order, whose values are—
    (1) 4, 4 and 5.
    (2) 9, 14, and 17.

21. Find the resultants of velocities in the directions of the sides of a square taken in order, whose values are—
    (1) 2, 2, 2, and 1.
    (2) 3, 2, 1, and 0.
    (3) 7, 5, − 3 and − 2.

22. A bullet has velocities of 400 northwards, of 300 eastwards, and of 1200 vertically upwards. Find its resultant velocity.

23. A ship is sailing at the rate of $11\frac{1}{4}$ miles an hour. A sailor climbs a mast 55 yards high in 24 seconds, find his rate of motion relatively to the surface of the earth.

24. Two men walk along two straight roads, which form a right angle, at the rate of 4 and $4\frac{1}{5}$ miles an hour. If they started from the intersection of the roads, how far apart will they be 5 hours afterwards?

25. The resultant of two velocities bisects the angle between them. Prove that the velocities are equal.

26. The resultant of two equal velocities is equal to either of them. Find the angle between the components.

27. A velocity of 12 is changed into a velocity of 5 at right angles to its former direction. Find the change of velocity.

28. A velocity of 13 has a horizontal component of 12. Find its vertical component.

29. Find the change of velocity in each of the following cases:—

(1) 4 north into 5 north.
(2) 4 north into 5 south.
(3) 1 east into $\sqrt{2}$ north-east.
(4) 1 east into 1 south.

30. A velocity represented by one side of an equilateral triangle is changed into that represented by another side taken in the opposite direction round the triangle. Find the change in magnitude and direction.

31. A velocity v has v_1 and v_2 for its rectangular components. Find v_2 in the following cases:—

(1) $v = 12\frac{1}{2}$, $v_1 = 7\frac{1}{2}$.
(2) $v = 29$, $v_1 = 21$.
(3) $v = 73$, $v_1 = 55$.

32. A velocity of 6 becomes one of 3 at an angle of 60° to its original direction. Find the change of velocity.

33. A particle moves with uniform acceleration during 3 seconds. Its initial velocity is 7 and its final velocity is 13. Find

(1) the average velocity.
(2) the space passed over.
(3) the acceleration.

GENERAL EXAMPLES.

34. A body starts with velocity 4, and has a uniform acceleration 1½ in the direction of motion. Find—

(1) the velocity after 10 seconds.
(2) the average velocity during that time.
(3) the space described in that time.

35. A body starts from rest with uniform acceleration 32. Find—

(1) how far it travels in 1 second.
(2) what distance is passed over in the second second.
(3) its velocity after 5½ seconds.
(4) the space described in 5½ seconds.

36. A body has an initial velocity 2½g and a uniform acceleration g. Find—

(1) its velocity after 4 seconds.
(2) the space described in that time.

37. A body starts from rest with uniform acceleration $g = 32{\cdot}2$. Find—

(1) the velocity after 5 seconds.
(2) the space described in 5 seconds.
(3) the velocity after n seconds.
(4) the space described in n seconds.
(5) after what time the velocity will be $4g$.
(6) after what time the distance travelled will be $18g$.

38. A body has velocity 29 and a uniform acceleration 3 opposite to the direction of motion. Find—

(1) the velocity after 7 seconds.
(2) the average velocity during that time.
(3) the space described in 7 seconds.
(4) after what time the body is at rest.
(5) the whole space described before it comes to rest.

39. A body has initial velocity ng and uniform acceleration g opposite to the direction of motion. Show—

(1) that its velocity after $n + r$ seconds is equal and opposite to its velocity after $n - r$.

(2) that its distances from its starting point at the above mentioned times are equal.

CHAPTER III.

KINETICS.

24. Preliminary.—In the preceding chapter we have considered *Motion* from a purely mathematical point of view. We have now to apply its results to the motion of material bodies under the action of force. The introduction to Dynamics is supplied by Newton in his *Principia*, by means of definitions and axioms. We shall follow his order as closely as possible.

25. Matter, Mass, Momentum.—MATTER *is that which occupies space.* It might also be defined as that which exerts force, or which can be acted on by force. A detached portion of matter is called a *Body.* A body so small that its dimensions may be neglected is called a *Particle.* We shall find that all that has been said in Kinematics about the motion of a point, is true in Kinetics of the motion of a Particle.

The Quantity of Matter in a body is called its MASS. It is convenient to take the *pound avoirdupois* as our unit of mass. The standard pound is a mass of platinum deposited in the Exchequer Office, and other masses are compared with it by weighing.

The Quantity of Motion in a body is called its MOMENTUM. It is measured by the product of the mass and the velocity of the moving body. *Unit momentum* is, therefore, the momentum of unit mass moving with unit velocity. Hence, if m be the number of pounds in the mass of a body, v its velocity, and M its momentum

$$M = mv \quad . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

Examples.—

1. Masses of 10 and 15 pounds have velocities 7 and 5 respectively. Compare their momenta.

Let m_1, m_2 be the masses.
" v_1, v_2, " their velocities.
" M_1, M_2, " " momenta.

Then by (13) $M_1 = m_1 v_1 = 70$

$$M_2 = m_2 v_2 = 75$$

$$\therefore \frac{M_1}{M_2} = \frac{70}{75} = \frac{14}{15}$$

2. The momentum of a mass m, moving with velocity v, is communicated to a mass m'. Find the velocity produced.

Let v' be the velocity produced.

Then by (13) $M = mv = m'v'$

$$\therefore v' = \frac{mv}{m'}$$

Examples for Exercise.—

1. A mass of 4 pounds has a velocity of 1000 feet per second. A mass of 100 has the same momentum, find its velocity.

2. A mass of 6 lbs. has velocity 12. If the mass were divided into 3 portions of 1, 2, and 3 lbs., what must be their respective velocities, in order that each portion may have $\frac{1}{3}$ of the original momentum of the whole mass?

26. Force.—As before, we define FORCE to be *any cause which tends to alter a body's state of rest, or of uniform motion in a straight line.*

A force is defined when its *direction*, its *magnitude*, and its *point of application* are known. It is therefore completely represented by a straight line, whose *direction* is that in which the force acts, whose *length* is proportional to its magnitude, and which is drawn *from its point of application.*

A uniform Force is measured by the momentum which it generates in a unit of time. Our unit of force is therefore that which produces unit of momentum in a second, that is, which acting for one second on unit mass generates unit velocity.

The earth's attraction for one pound of matter, at or near the level of the sea, in any part of the British Islands, gives it in one second a velocity of very nearly 32·2. (See § 37.) The weight of a pound is therefore 32·2 times the unit force, that is, it is a force of 32·2.

Examples.—

1. A force 12 acts on a mass of 4 lbs. for 6 seconds, find the velocity produced.

In each second a momentum 12 is generated, and therefore in 6 seconds a momentum 72

$$\begin{aligned} &\text{Or} & mv &= 6 \times 12 = 72, \\ &\text{But} & m &= 4, \\ && \therefore v &= \frac{72}{4} = 18. \end{aligned}$$

2. How long must the earth's attraction act on a mass of 3 lbs. to produce a velocity 64·4?

Here we must observe that since the earth attracts every part of the 3 lbs., the force applied is 3 × 32·2 or 96·6.

This number, therefore, represents the momentum generated in each second.

A mass 3 moving with velocity 64·4 has a momentum of 3 × 64·4 or 193·2.

The velocity required is therefore produced in two seconds.

Examples for Exercise.—

1. What force must act on a mass of 5 lbs. for 2 seconds to give it a velocity of 13?

2. What time must a force 6 act on a mass 4, in the direction of its motion, to increase its velocity from 8 to 20?

27. **Work.**—*A force does* WORK *when the body on which it acts moves in the direction of the force.* Thus, when a stone falls vertically downwards, gravity does work.

When a stone slides or rolls down a hill, its velocity may be resolved into two components, a horizontal and a vertical.

The vertical component is in the direction of gravity, and therefore gravity does work in this case also.

When a body is raised, it is acted on by a force equal and opposite to that of gravity. This force does work, and the work done is said to be done against gravity.

The work done by a uniform force acting on a body is measured by the product of the force and the distance through which the body moves in the direction of the force.

The scientific unit of work is that done by unit force acting through unit space. The unit practically used in this country is the *foot-pound,* or the work required to raise the weight of one pound through one foot. As we have already seen, the weight of one pound is a force of 32·2, and hence a foot-pound is equal to 32·2 scientific units of work.

Example. — A weight of 4 pounds is allowed to slide down an inclined plane whose height is three feet. Express in foot-pounds the work done by gravity.

Since gravity acts vertically downwards, we must multiply the weight of the body in pounds by the distance which the body moves vertically downwards. The work done is therefore 12 foot-pounds.

Example for Exercise.—How much work is done against gravity by a man who weighs 14 stones, in climbing a mountain 3000 feet high?

28. Energy.—ENERGY *is the power of doing work.* It is of two kinds, *Kinetic* and *Potential.*

Kinetic Energy is the power which a moving body has of doing work in virtue of its motion. It is measured by half the product of the mass and the square of the velocity. Thus a mass m moving with velocity v has Kinetic Energy $\frac{1}{2}mv^2$.

When a body is allowed to move freely under the action of force, the Kinetic Energy gained is the equivalent of the work done. Thus the Kinetic Energy of a falling body is the equivalent of the work done by gravity on it. If, on the other hand, a body is constrained to move against a force, the work done takes the form of *Potential Energy.*[*]

Thus, a weight requires work to raise it to a height, but the raised weight is a store of energy which can be made use of at pleasure.

Bodies have, therefore, Potential Energy in virtue of their position with regard to other bodies which exert force on them.

Examination on Sections 24—28.

1. Define matter. What is meant by a particle?

2. Define mass. What is the British unit of mass?

3. Define momentum. How is the momentum found when the mass and velocity of a moving body are given?

4. Define force. What are the three elements which specify a force? Show that a straight line completely represents a force.

5. How is a force measured? Show that the weight of 1 lb. is a force of 32·2.

6. When is a force said to do work? How is the work done by a force measured?

7. What is meant by a foot-pound? Compare it with the scientific unit of work.

8. Define energy. What are the two kinds of energy?

9. What is kinetic energy? Give an expression for the kinetic energy of a mass m moving with velocity v.

10. When does work done on a body take the form of potential energy?

29. The Laws of Motion.—We now proceed to the axioms of the subject, which are called the Laws of Motion, and are here translated from the Latin of the *Principia*.

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[*] This is only true in the case of forces which act equally and in the same direction, if the motion of the body be reversed. Thus it is true if the force be gravity, but not true if the force be friction.

The axioms of Dynamics are self-evident to those who have a sound experimental knowledge of physical phenomena. They are proved to be true by the fact that the results, which are deduced from them, are in accordance in every case with the observed event.

30. First Law of Motion.—*Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by forces to change that state.*

If we propel a curling-stone along a horizontal surface of ice, its velocity gradually becomes less. This is due entirely to two causes, the roughness of the ice and the resistance of the air. The smoother the ice and the rarer the atmosphere, the more nearly uniform is the velocity of the curling-stone. Hence we infer that if we could obtain perfectly smooth ice and a perfect vacuum, the velocity of the curling-stone would be uniform.

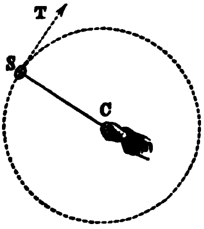
31. Inertia.—*The property of matter in virtue of which it tends to maintain its state of rest or of uniform motion in a straight line is called its* INERTIA.

The first law of motion asserts the existence of inertia, and that force is required to change a body's state of rest or motion.

Thus the inertia of our bodies is the cause of the jerks we receive when sitting in a carriage which is suddenly set in motion, or suddenly stopped. In the first case our state is one of rest, and tends to remain so, in the second it is one of motion, and tends to continue so.

32. The so-called Centrifugal Force.—The first law of motion shows us that, whenever a body moves in a curve, it does so because of some force which continually changes its direction, and thus prevents it from continuing to move in a straight line.

As an example, consider the case of a stone S tied to a string SC, and swung in a circle round the hand C. At any instant the stone is moving in the direction of the tangent ST, and would continue to move in that line were it not prevented from doing so by the tension of the string. If, however, the string break, the stone will fly off along the tangent.



In this case, the tension of the string is the force which, at each instant, prevents the stone from continuing its natural motion in a straight line. It was formerly supposed that this force was rendered necessary by a centrifugal force, called into action by the fact of the body's moving in a curve; and the name is still generally applied to the tendency such a body has to fly off at a tangent, and *thus* increase its distance from the centre. The term is doubly misleading. First, it is not an effect of *force*, but a result of the inertia of the moving body; secondly, it is not, properly speaking, *centrifugal*, for the tendency is not to motion along CS produced, but along ST.

The following are examples of bodies which move in curves owing to the action of some force.

The earth in its annual motion round the sun has its direction continually changed by the sun's attraction. The particles of a fly-wheel are forced by cohesion to move in a circle; if the velocity of the fly-wheel is made so great as to overcome the force of cohesion, the wheel will fly asunder. When a wet mop is trundled, the adhesion of the drops of water is not sufficient to prevent their flying off at a tangent.

33. The Measurement of Time.—The first law of motion gives us a means of measuring time. It tells us that every body, unacted on by force, passes over equal spaces in equal times. It follows from this that a rotating body, whose rotation is unresisted, turns through equal angles in equal times; and hence we can define equal times as those in which the earth turns through equal angles.

Examination on Sections 29—33.

1. Enunciate the first law of motion, and illustrate it by an example.
2. Define inertia.
3. What is meant by centrifugal force? Show that the term is a misleading one.
4. Show that the first law of motion enables us to define equal portions of time.

34. Second Law of Motion.—*Change of Momentum is proportional to the force which causes it and is in the direction of the straight line in which the force acts.*

In other words, a given force acting on a body for a given time always produces the same change of momentum, whatever be the mass of the body on which it acts, and whether that body be at rest or in motion in any direction with any velocity.

35. The Measurement of Force.—The second law of motion gives us a means of comparing different forces. Apply different forces for equal times to the same body at rest and observe the velocities produced. Since the mass is the same in each case, the momenta generated are proportional to the velocities, but, by the second law of motion, the forces are proportional to the momenta, and therefore to the velocities.

Algebraical Example.—Forces f_1 and f_2 act on a mass m for equal times and give it respectively the velocities v_1 and v_2; compare the forces.

Since the forces are measured by the momenta, we have—

$$f_1 = mv_1,$$
$$f_2 = mv_2,$$
$$\therefore \frac{f_1}{f_2} = \frac{mv_1}{mv_2} = \frac{v_1}{v_2},$$ or the forces are proportional to the velocities.

36. The Measurement of Mass.—The second law also enables us to compare masses. For if equal forces act for equal times on different masses, the velocities produced are inversely as the masses.

Again, if different forces be applied for equal times to these masses, so as to give them the same velocities, then the forces must be proportional to the masses. Thus we see that a mass may be measured by the force required to give it a certain velocity. In other words—a mass is measured by its inertia.

Example.—The same force is applied for one second to masses of 4, 6, and 8. When applied to 4, it gives it a velocity of 12. Find the force. What velocities will it give the other masses?

Here $f = mv' = 48$
Therefore when $m = 6, v = 8.$
And when $m = 8, v = 6.$

Thus the velocities produced are inversely as the masses.

Example for Exercise.—Two forces f_1 and f_2 applied for equal time to masses 9 and 12, give the same velocity. Find f_1, if $f_2 = 8$.

37. The Force of Gravity.—The great law of gravitation announced by Sir Isaac Newton is, *Every particle in the universe attracts every other with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses, and inversely as the square of their distance from each other.*

A particle of matter near the surface of the earth attracts, and is attracted by, all the particles of which the earth is composed. The resultant (see § 41) of all these forces is a single force tending to move the particle towards the centre of the earth, and called the force of gravity, or the weight of the particle.

It follows from the law of gravitation that this force is proportional to the mass of the body, and varies inversely as the square of its distance from the centre of the earth. Thus a body weighs less at the top of a hill than at the bottom, and, owing to the shape of the earth, weighs less at the equator than the poles. But this variation is so slight that in the case of small distances it may be neglected, and we may regard the force of gravity as constant. We have already seen that the attraction of the earth on a mass of one pound is nearly equal to 32·2 units of force in the British Islands. It is convenient to represent this number by the letter g. Thus $g = 32{\cdot}2$. If the mass of a body be m pounds, then its weight is mg. Thus bodies of the same mass have the same weight *at the same place.*

38. Falling Bodies.—We are now able to investigate the motion of bodies falling under the action of gravity.

The earth attracts a mass m, with a force mg, and therefore, if the mass be acted on by gravity alone, it will receive a downward momentum, mg, in each second, and thus its downward velocity will be increased in each second by g units of velocity. We have therefore uniform acceleration in the direction of motion, the case discussed in § 22, substituting g for α in equations (9) and (10), and we obtain

$$v = V + gt \quad . \quad . \quad . \quad . \quad . \quad (14)$$
$$s = Vt + \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad (15)$$

the equations of motion of a falling body, with an initial velocity V, downwards. If the body fall from rest, $V=0$ and the equations become

$$v=gt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$
$$s=\tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Examples.—

1. A mass of 6 lbs. falls from rest. How far will it fall in 10 seconds, and what will be its momentum after that time?

By (17) $s = \frac{1}{2}gt^2$
$= \frac{1}{2} \times 32{\cdot}2 \times 100 = 1610$ feet.

By (16) $v = gt$
$= 32{\cdot}2 \times 10 = 322$

$\therefore$ its momentum $mv = 6 \times 322 = 1932$.

2. A stone is thrown down a well with velocity 50. If it reach the bottom in 2 seconds, how deep is the well?

By (15) $s = Vt + \frac{1}{2}gt^2$
$= 50 \times 2 + \frac{1}{2} \times 32{\cdot}2 \times 4 = 164{\cdot}4$ feet.

3. A body falls from rest. In what time will it fall 100 feet?

By (17) $s = \frac{1}{2}gt^2$

$$\therefore \quad t^2 = \frac{2s}{g} = \frac{200}{32{\cdot}2} = 6{\cdot}21 \text{ approximately}$$

$$\therefore \quad t = \sqrt{6{\cdot}21} = 2{\cdot}5 \text{ seconds, approximately.}$$

Examples for Exercise.—

1. A stone weighing 4 lbs. is dropped from a balloon, and reaches the earth after 18 seconds. Find the momentum with which it strikes the earth, and the height of the balloon.

2. With what force must a bullet of 2 ounces weight be propelled downwards in order that it may fall 1000 feet in two seconds?

3. A well is 559 feet deep. Find approximately the time which will elapse between dropping a stone into the well and hearing the sound of the splash. (Take the velocity of sound as 1118).

39. Bodies thrown up.—If a body be thrown vertically upwards, gravity causes a uniform acceleration opposite to the direction of motion, the case discussed in § 23. Substituting g for a in equations (11) and (12) we obtain—

$$v = V - gt \quad . \quad . \quad . \quad . \quad . \quad (18)$$
$$s = Vt - \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Example.—A ball is thrown vertically upwards with velocity 50. Find how high it will rise, and when and with what velocity it will return to the ground.

We observe that as the ball rises the velocity v gradually diminishes, and that at the instant of the ball's reaching its highest point it has no velocity at all, or

$$v = 0.$$

At this instant s has reached its maximum value. After this instant the velocity is downwards or v is negative, and s gradually decreases until the ball again reaches the ground, when

$$s = 0.$$

(1.) To find how high the ball will rise. At the highest point $v = 0$ and therefore—

By (18) $\quad V - gt = 0$

$$\therefore \quad V = gt,$$
$$\therefore \quad t = \frac{V}{g} = \frac{50}{32 \cdot 2} = 1 \cdot 55, \text{ approximately.}$$

By (19)
$$\begin{aligned} s &= Vt - \tfrac{1}{2}gt^2 \\ &= 50 \times 1 \cdot 55 - \tfrac{1}{2} \times 32 \cdot 2 \times (1 \cdot 55)^2 \\ &= 38 \cdot 7 \text{ feet, approximately.} \end{aligned}$$

(2.) To find when the ball will return to the ground.

At this instant—

$$s = 0.$$

And ∴ by (19) $\quad Vt - \tfrac{1}{2}gt^2 = 0.$

$$\therefore \quad V - \tfrac{1}{2}gt = 0.$$
$$\therefore \quad V = \tfrac{1}{2}gt.$$
$$\therefore \quad t = \frac{2V}{g} = \frac{100}{32 \cdot 2} = 3 \cdot 1 \text{ seconds, approx.}$$

(3.) To find the velocity with which the ball reaches the ground—

By (18) $v = V - gt.$

But at this instant $t = \frac{2V}{g}$

$$\therefore \quad v = V - g \times \frac{2V}{g} = V - 2V = -V.$$

Or the ball returns to the ground in the same time that it took to rise to its highest position, and reaches the ground with the same velocity as that with which it was thrown up.

Example for Exercise.—With what velocity must a body be thrown vertically upwards that it may remain in the air for 4 seconds, and how high will it be at the end of each second?

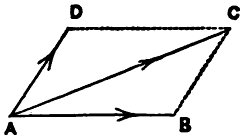
40. Parabolic path of a Projectile.—When a body is projected in any direction other than vertical, its position at any moment is found by considering its velocity as resolved in two directions, vertical and horizontal. Its vertical velocity is altered by gravity in the manner described in the last two paragraphs, but its horizontal velocity remains constant, as the body is not acted on by any horizontal force. Combining these results, it may be proved geometrically that the body describes a parabola, the curve which is represented in the figure illustrating § 3.

41. The Composition of Forces.—It follows from the second law of motion that, when any number of forces act simultaneously on a particle, each of them produces the same velocity that it would have produced if it had acted singly on the body at rest. The resultant motion is therefore the resultant of these velocities.

A force which, acting singly, would have produced this velocity in the particle is called the Resultant Force, and the several forces are called Components.

Parallelogram of Forces.—*If two forces acting simultaneously on a particle be represented by adjacent sides of a parallelogram,* *the resultant force is represented by the diagonal passing through their intersection.*

Proof.—Let AB, AD represent two forces acting simultaneously on a particle at A. Then since forces are measured by the momenta they produce in unit of time, AB and AD are proportional to the momenta, and therefore to the velocities, which these forces acting separately would produce in the particle at A.



But by the parallelogram of velocities, AC is the resultant velocity, and therefore AC represents the resultant force.

In the same manner, by substituting force for velocity, all the propositions regarding the composition and resolution of velocities, proved in §§ 13, 14, 15, 16, 17, and 18, are proved to be equally true of forces.

Examples.—

1. Forces which are represented in magnitude and position by all but one of the sides of a polygon taken in order act simultaneously on a particle; find their resultant.

Since the sides of the polygon represent the forces, they also represent the velocities which the forces would produce acting separately. By § 14 the resultant of these velocities is the remaining side taken in the opposite direction. The resultant force is therefore also represented by that side.

2. Forces 5 and 12 act on a particle at right angles, find their resultant.

Let f be the resultant force. Then applying § 15 to forces we have—

$$f^2 = 5^2 + 12^2 = 25 + 144 = 169.$$
$$\therefore f = 13.$$

Examples for Exercise.—

1. Prove that if three forces are represented by three sides of a triangle taken in order, they will produce equilibrium.

2. Find the resultant of forces $3\frac{1}{2}$ and 4 at an angle of 60°.

3. A vertical force is equal to 12. Find its component along a line which makes an angle of 30° with the horizon.

Examination on Sections 34—41.

1. Enunciate the second law of motion.

2. Show that if different forces are applied for equal times to equal masses the velocities generated are proportional to the forces.

3. How is mass measured?

4. Give the law of universal gravitation. What is the average force of gravity in the British Islands?

5. Find the equations of motion of a body falling from rest under the action of gravity.

6. If a body is thrown up with velocity V, show that its velocity after t seconds is $V - gt$, and that its distance from the ground at the same time is $Vt - \frac{1}{2}gt$.

7. What is meant by the resultant of any number of forces?

8. Assuming the parallelogram of velocities, prove the parallelogram of forces.

42. The Third Law of Motion.—*To every action there is always an equal and contrary reaction.*

When one body presses another, it is pressed by that other with an equal force in the opposite direction. A stone resting on a table causes pressure on the table in consequence of its weight, but the table exerts an equal and opposite pressure on the stone, and this pressure, acting on the stone, balances the force of gravity, and keeps the stone at rest.

Again, when one body attracts another, it is attracted by that other with equal force in the opposite direction. Thus the moon exercises on the earth the same attraction that the earth exercises on the moon, but in the opposite direction.

43. Recoil.—The third law of motion enables us to investigate the recoil of a gun when a projectile leaves it. Since action and reaction are equal and opposite, the momentum acquired by the gun is the same as that acquired by the projectile, or, if M, m be their masses, and V, v their velocities,

$$MV = mv \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Example.—A gun weighing 100 tons discharges a shot weighing 1000 lbs. If the initial velocity of shot be 1250, find that of the recoil of the gun.

From (20) $V = \frac{mv}{M} = \frac{1000 \times 1250}{100 \times 20 \times 112} = 5\frac{65}{112}$

Example for Exercise.—A bullet weighing 2 ounces is discharged from a rifle weighing 10 lbs., and the initial velocity of the recoil is observed to be 14, find the velocity with which the bullet leaves the rifle.

44. Conservation of Energy.—In a scholium appended to the third law of motion, Newton gives another kind of actions and reactions subject to the third law. This scholium has been converted into the following form. *Work done on any system of bodies has its equivalent in work done against friction, molecular forces, or gravity, if there be no acceleration ; but if there be acceleration, part of the work is expended in overcoming the resistance to acceleration, and the additional kinetic energy developed is equivalent to the work so spent.* (Thomson & Tait's *Natural Philosophy*, § 269).

From this scholium the principle of *Conservation of Energy* follows. This principle asserts that the amount of energy in the universe is constant, that it may be transmitted from one body to another, or transformed from one species to another, but can neither be increased nor diminished in quantity. Besides energy of visible motion, Heat, Light, Sound, and Current Electricity are all recognised as forms of Kinetic Energy. Gravity, Chemical Attraction, Electric, and Magnetic Attraction depend on Potential Energy.

The behaviour of a common pendulum illustrates the principle of Conservation of Energy. When the pendulum is in the middle of its swing, it possesses a certain amount of kinetic energy. As it approaches its turning point its energy of motion gradually diminishes, and finally disappears; but the bob of the pendulum has been raised a small distance against the force of gravity, and has thus acquired a certain amount of potential energy, which is in its turn reconverted into kinetic energy. These quantities of potential and kinetic energy would be an exact equivalent for each other, and the pendulum would continue to swing through equal distances for ever, were it not that during each swing a certain amount of the energy is required to set the air in motion, and to overcome friction. Thus the energy of the pendulum is gradually transmitted to the air around, and in course of time is all transformed into that form of kinetic energy which we call heat.

Examination on Sections 42—44.

1. Give Newton's third law of motion.

2. Show how the third law of motion applies to a book resting on a table, and to a horse pulling a canal-boat.

3. Give the scholium to Newton's third law.

4. Explain what is meant by Conservation of Energy.

5. Illustrate the principle of Conservation of Energy by means of (1), the swinging of a pendulum; (2), a stone thrown on to the roof of a house.

6. Name several forms of energy, and give instances of transformation of energy from one form to another.

General Examples in Kinetics.

1. Find the momentum of each of the following systems.

(1). Masses of 4, 7, and 10 pounds moving in the same direction, with velocities 12, 10, and 6 respectively.

(2). A mass of 10 pounds, with velocity 6, and a mass of 12 pounds moving in the opposite direction with velocity 5.

2. A mass of 30 pounds has a velocity of 100 in a direction making an angle of 60° with the horizon. Find its horizontal momentum.

3. How long must gravity act on a falling body to give it a velocity of 161?

4. How much work is done against gravity by a man weighing 12 stones in climbing a mountain 4000 feet high?

5. How much work is done by the same man in walking a mile along a level road?

6. In Attwood's machine the weight of one ounce is employed to move a mass of two pounds. Find the velocity of the mass after one second.

7. How far will a stone fall from rest in 5 seconds, and what velocity will it have at the end of that time?

8. A rifle ball is discharged horizontally from a position 1000 feet above the level of the sea, what will be its height 4 seconds after the discharge?

9. How long will a body falling from rest take to acquire a velocity of 80 feet per second?

10. How far will a body fall from rest in $3\frac{1}{2}$ seconds?

11. A stone dropped from the top of a cliff is observed to reach the bottom in $6\frac{1}{2}$ seconds. Find the height?

12. A body, which has fallen from rest, reaches the ground with a velocity of 1127 feet; find how long the body has been falling, and the distance it has travelled.

13. A weight of 10 lbs., resting without friction on a horizontal table, has a constantly acting horizontal force of 2 lbs. applied to it, what will be the velocity after 1 second?

14. With what velocity must a ball be thrown up, in order to return to the hand after the lapse of 1 second?

15. A body is projected upwards with a velocity of 161 feet per second, how high will it rise?

16. A stone thrown vertically upwards strikes the ground after an interval of 10 seconds, with what velocity was it projected, and to what height did it rise?

17. A body is projected vertically upwards with a velocity of 80 feet. Where will it be after 2, 4, 6 seconds respectively?

18. A body is thrown upwards with a velocity of 250. Find its velocity after 7 seconds.

19. (1) An arrow is shot vertically upwards with a velocity of 100 feet in a second. How long will it be before it reaches the ground again?

(2) Another arrow, half the mass of the former, is shot vertically upwards with the same momentum. How long will it be before it reaches the ground?

20. A projectile is discharged in a slanting direction upwards. If there had been no gravity it would have reached an altitude of 1000 feet above the earth's surface at the end of one second. Find its actual altitude at that moment?

21. Let AB and AC represent in magnitude and direction two equal forces acting at A, the angle between their directions being 50°. Draw a line which shall represent in direction and magnitude a force in equilibrium with these forces; and state the angle which its direction makes with AB.

22. Find the resultant of the following forces:—5 northwards, 5 eastwards, 1 westwards, and 2 southwards.

23. Two forces, in magnitude 3 and 4, act at the same point in directions at right angles to each other. Draw their resultant and find its magnitude.

24. If a particle be urged towards the east with a force of 10 lbs., and towards the north with an equal force, what is the joint effect of the force, and in what direction will the particle be urged?

25. A body is pushed simultaneously by a force of 3 lbs. acting from east to west, a force of 4 lbs. acting from north to south, and a force of 12 lbs. acting vertically downwards. What is the resultant force that acts upon the body?

26. A gun weighing 8 lbs. avoirdupois discharges a one-ounce ball with the velocity of 1000 feet per second. What will be the velocity of recoil of the gun?

27. A shot weighing 30 lbs. is fired from a gun weighing 3 tons, and leaves the gun with a velocity of 1120 feet per second. Find the velocity of the gun's recoil.

28. A shell bursts into two fragments, whose weights are 12 and 20 pounds. The first travels onwards with a velocity of 700

and the latter with a velocity of 380. What was the momentum of the shell when the explosion took place?

29. A shell weighing 20 lbs. explodes when moving with a velocity of 600; at the moment of explosion one-third of the shell is reduced to rest. Find the momentum of the other two-thirds.

30. A shell in motion, with a velocity of 80 feet per second, bursts into two equal pieces. If one of them continue its path with a velocity of 120 feet, what will be the motion of the other?

31. A force of 97 lbs. is resolved into two component forces at right angles. If one component is 65 lbs., find the other.

32. Find the magnitude of the resultant of two forces of 13 and 35 lbs. when the angle between their directions is 60°.

33. Find the velocity which a heavy body will acquire by falling freely through a height of $3\frac{1}{23}$ feet.

34. Two forces act at right angles, find their resultant when their values are—

(1) 48 and 55.
(2) a and b.
(3) $a^2 - b^2$ and $2ab$.

35. A body is projected with a velocity of 10 feet per second and is brought to rest by a uniform force after passing over 10 feet in a straight line. Find the magnitude of the force.

36. Two forces act on a particle. When their direction is such that their resultant is greatest, their resultant is 12; when their resultant is least, it is 6. Find the forces.

37. R is the resultant of two forces P and Q. If R be at right angles to P and equal to half Q, find the angle at which P and Q act.

38. The resultant of two forces is represented by the diameter of a circle. If one force be represented by a chord through one extremity of the diameter, prove that the other force will be represented by the chord drawn through the same extremity at right angles to the first chord.

CHAPTER IV.

STATICS.

45. Equilibrium of a Particle.

Two Forces. *Two equal and opposite forces, acting on a particle, are in equilibrium.*

By the second law of motion (see § 41), the two forces give the particle two equal and opposite velocities, and therefore its velocity is unchanged (§ 11 Cor.).

Three Forces. *Three forces, acting on a particle, which are represented by the three sides of a triangle taken in order, are in equilibrium.*

By the second law of motion the three forces give the particle three velocities, which are represented by the sides of the same triangle taken in order, and therefore its velocity is unchanged (§ 13).

This proposition has two important converse propositions, which we proceed to prove;—

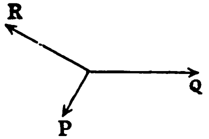
(i.) *If three forces, acting on a particle, are in equilibrium, they may be represented by three sides of a triangle taken in order.*

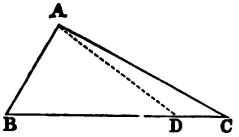
For, if lines representing two of the forces be placed so as to form two sides of a triangle taken in order, and the triangle be completed, the third side of the triangle must, by the preceding proposition, represent a force in equilibrium with the first two. It must therefore represent the third force.

(ii.) *If three forces, acting on a particle, are in equilibrium, and if their directions are those of the sides of a triangle taken* *in order, then, if one of the sides of the triangle be taken to represent the force to which it is parallel, the other sides will, on the same scale, represent the other forces.*

Let the forces P, Q and R be in equilibrium, and let the sides of the triangle ABC be parallel to their directions. Then if AB be taken to represent P, BC will represent Q, and CA will represent R, on the same scale.

For, if BC does not represent Q, let BD represent it.





Join AD. Because ABD is a triangle, the force represented by DA will be in equilibrium with those represented by AB and BD. Therefore DA will represent R. But this is impossible, since CA is parallel to the direction of R.

Therefore BC represents Q.

Similarly CA represents R.

Any number of Forces. *If any number of forces, acting on a particle, be represented in magnitude and direction by the sides of a polygon taken in order*, then by the polygon of velocities (§ 14) the velocity of the particle is unchanged, and therefore *the forces are in equilibrium.*

Examples.—

1. Forces of 6, 8, and 10 acting on a particle are in equilibrium, prove that the directions of the first two form a right angle.

Since the three forces are in equilibrium, a triangle can be formed whose sides represent the forces in magnitude and direction. The sides of this triangle will therefore be 6, 8, and 10.

But $10^2 = 100 = 64 + 36 = 8^2 + 6^2$.

∴ The triangle is right angled (Euclid I., 48).

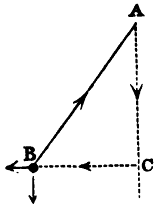
∴ The forces 6 and 8 act at right angles.

2. Three forces, acting on a particle, are in equilibrium, prove that they lie in one plane.

As before, a triangle can be formed whose sides represent the forces in magnitude and direction.

∴ The three forces all lie in the plane of the triangle.

* 3. A weight of 8 lbs. hangs by a chain 20 feet long, and is pulled out by a horizontal force to a distance of 12 feet from the vertical through the point of support. Find the tension of the chain.



Let AB be the chain. Draw AC vertically downwards, and from B draw BC at right angles to AC. Then $AB = 20$, $BC = 12$, and ABC is a right angled triangle.

$$\therefore AC^2 = AB^2 - BC^2 = 256.$$

$$\therefore AC = 16.$$

The weight at B may be considered as a particle. It is kept in equilibrium by three forces. These forces are the tension of the string acting in the direction BA, the weight, which acts vertically downwards, and the horizontal force. These three forces are in the direction of the sides of the triangle BAC taken in order. If we take AC to represent the weight, CB will represent the horizontal force, and BA the tension of the string.

Thus a line of 16 feet represents a force of 8 lbs.
,, 2 feet ,, ,, 1 lb.
,, 20 feet ,, ,, 10 lbs.

But AB, which represents the tension of the chain, is 20 feet, and therefore the tension is 10 lbs. (That is, it is the same as if the chain hung vertically downwards, and supported a weight of 10 lbs.)

Examples for Exercise.—

1. Three equal forces are in equilibrium, find the angles between them.

2. A weight of 100 lbs. is suspended by two strings, each of

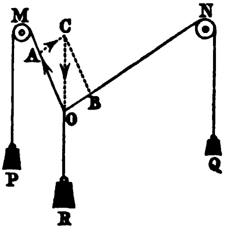
---

* In this and the following chapters some demonstrations occur which may possibly present difficulty to those students who are less familiar with mathematical proofs. Should this be the case, the student is advised to omit these in his first reading of the book. They have, on this account, been printed in a smaller type.

which makes an angle of 30° with the horizon. Find the tension of each string.

3. Show that, if three forces acting at a point are in equilibrium, any two of these forces are together greater than the third.

46. Experimental verification of the Parallelogram and Triangle of Forces.—The laws which regulate the equilibrium of three forces may be verified in the following way. Three weights, P, Q and R, are attached by three strings knotted together at O. Two of the strings, OP and OQ, are placed over two pulleys M and N, and the third string, OR, is allowed to hang vertically downwards. If the weights are such that equilibrium is possible,* the system will come to rest in some such position as that shown in the figure.



The effect of the pulleys at M and N is simply to change the direction of the strings without altering their tensions (§ 65). We have therefore the forces P and Q acting at O in the direction OM and ON, and these are in equilibrium with the force R in the direction OR.

Mark off, on OM and ON, lengths OA and OB, to represent the forces P and Q. Complete the parallelogram OACB. Then OC is the resultant of OA and OB. It will be found that OC is vertical, and that its length represents the weight R on the same scale as that on which OA and OB represent P and Q. Thus the parallelogram of forces is experimentally verified.

We also observe that, in the triangle OAC, OA, AC, and CO represent in magnitude and direction the three forces P, Q and R, and thus the triangle of forces is also experimentally verified.

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* See last Example for Exercise above.

47. A Rigid Body. *A Rigid Body is a collection of material particles so united that their relative positions do not change.*

No body is perfectly rigid, but all solids are approximately rigid, when the forces which act on them are not too great.

48. Equilibrium of a Rigid Body.

Two Forces.—*Two equal forces, acting on a rigid body at rest in opposite directions along the same straight line, maintain equilibrium.*

Let P act at A and Q at B in opposite directions along the straight line AB. The force P tends to move the particle at A, and to move with it the whole mass of the body. The force Q tends to move the particle at B in the opposite direction, and along with it the whole mass of the body. But owing to the rigid connection of A and B these two forces neutralise each other, and equilibrium is maintained.

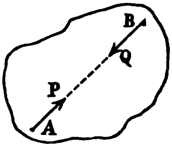


Figure 1.

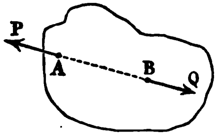


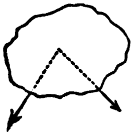
Figure 2.

This fact may be verified by experiment.

If P act at A in the direction AB, then it is neutralised by the equal and opposite force Q, at whatever point in the line AB, Q is supposed to act. Hence, if at B two equal and opposite forces, each equal to P and in the direction of the line AB be applied, one of these forces neutralises P, and the other acting at B is equivalent to P at A. This is often called the principle of the *transmission of force.*

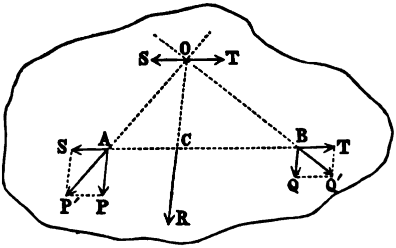
Since the two forces P and Q are in equilibrium, if any number of forces act on a body, we may suppose such a pair of forces as P and Q to act along with them, and the result will be unaltered.

Forces whose Lines meet.—If the lines of action of two or more forces meet in a point, we may suppose the forces to act at that point, and the conditions of equilibrium are the same as those already investigated for a particle.



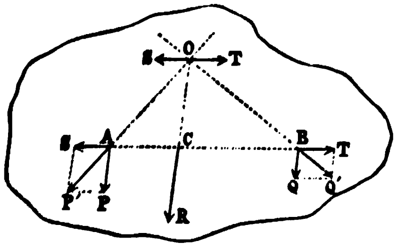
When the directions of the forces do not all pass through a point, the problem is more complicated, and the only case of this kind, which will be considered in this treatise, is that of parallel forces.

49. Resultant of Two Parallel Forces.—Let P and Q be two parallel forces acting on a body, and let A and B be two points in their lines of action. Join AB. At A and B apply two equal and opposite forces S and T, acting in opposite directions along AB. Then by § 48 the effect of P and Q will be unaltered. Thus the four forces P, Q, S, and T are equivalent to the two forces P and Q. Let P′ be the resultant of S and P, and let Q′ be the resultant of Q and T, and let the directions of P′ and Q′ meet in O; P′, and Q′ may be considered as acting at O.



From O draw OC parallel to the direction of P and Q, and meeting AB in C.

Substitute for P′ at O the equivalent forces P and S, and for Q′ the equivalent forces Q and T.



Then S and T, being equal and opposite, may be removed, and there remain P and Q acting in the direction OC.

Call this resultant R, then—

$$R = P + Q \quad . \quad . \quad . \quad . \quad (22)$$

and acts at C, parallel to the direction of P and Q.

To find the position of C.

The three forces P, P′, and S are parallel to the sides of the triangle OAC. If, for a moment, we consider OC to represent P, AC will, on the same scale, represent S.

$$\therefore \frac{OC}{AC} = \frac{P}{S}$$

Multiplying both sides of this equation by S × AC, we obtain—

$$S \times OC = P \times AC. \quad . \quad . \quad . \quad . \quad (a)$$

Also the three forces Q′, Q, and T are parallel to the sides of the triangle OCB. If, therefore, OC represent Q, BC will represent T.

$$\therefore \frac{OC}{CB} = \frac{Q}{T}$$

Multiplying both sides by T × BC we obtain—

$$T \times OC = Q \times BC. \quad . \quad . \quad . \quad . \quad (b)$$

Combining the equations (*a*) and (*b*), and remembering that S = T, and therefore S × OC = T × OC, we obtain the equation—

$$P \times AC = Q \times BC. \quad . \quad . \quad . \quad . \quad (23)$$

If we divide both sides of this equation by Q × AC, it takes the form—

$$\frac{P}{Q}=\frac{BC}{AC} \quad . \; . \; . \; . \; . \; . \quad (24)$$

We see from this equation that the forces P and Q are inversely proportional to their distances from C.

Examples.—

1. Parallel forces 5 and 3 act at two points 8 inches apart. Find their resultant and its point of application.

Let P and Q be the forces, A and B their points of application.

$\therefore$ By (22) $R=P+Q=8$.
By (23) $P \times AC=Q \times BC$.

Let $AC = x$ inches, then $BC=8-x$ inches; substituting these values for P and Q, equation (23) becomes—

$$5x=3\ (8-x),$$
$$\text{Or } 5x=24-3x.$$
$$\therefore 8x=24.$$
$$\therefore x=3.$$

Thus the resultant force is 8, and acts at a point C, 3 inches from A along AB.

2. P and Q are equal parallel forces, find their resultant and its point of application.

By (22) $R=P+Q=2\,P$.
By (23) $P \times AC=Q \times BC$.
But $P=Q$. $\therefore AC=BC=\frac{1}{2}AB$.

Examples for Exercise.—

1. Weights of 5 and 7 lbs. are hung at the extremities of a rod 2 feet long. Find their resultant and its point of application.

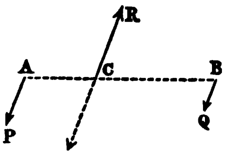
2. Forces of 3 and 9 lbs.* act in parallel directions at two points 20 inches apart. Find their resultant and its point of application.

3. Three forces in one plane are in equilibrium. Show that they must either be parallel forces, or forces whose directions meet in a point.

---

* By a force of 3 lbs. is meant a force equal to the weight of 3 lbs. (See § 26.)

50. Equilibrium of Three Parallel Forces.—Since in the figure of § 49, R, acting at C, is the resultant of the parallel forces P and Q, if we apply at C a force equal and opposite to R, it will be in equilibrium with P and Q.



Thus the three forces, P, Q, and R in the adjoining figure are in equilibrium.

Since any one of these three forces is equal and opposite to the resultant of the other two, we may apply the equations of § 49 to find the resultant of parallel forces acting in *opposite* directions. The following examples will serve to explain how this is done.

Examples—

1. P and R are parallel forces of 3 and 4 lbs. acting in opposite directions at two points, A and C, 3 inches apart. Find a third force which will produce equilibrium.

Let Q be the third force, and B its point of application.

Then by (22) $R=P+Q$.

$\therefore Q=R-P=4-3=1$.

Also by (23) $P \times AC=Q \times BC$.

Let $BC = x$, and substituting their values for P, Q and AC, this equation becomes—

$$3 \times 3=1 \times x,$$

Or $$x = 9.$$

Thus Q is a force of 1 lb., acting parallel to P at a point B, 9 inches from C, in AC produced.

The resultant of P and R is therefore a force of 1 lb. acting at B in the direction of the greater force R.

2. Find the resultant of two opposite parallel forces 7 and 4, acting at two points 6 inches apart.

As before, let $P=4$, $R=7$, $AC=6$, $BC=x$.

Then by (22) $Q=R-P=3$.

And by (23) $P \times AC=Q \times BC$.

Or $4 \times 6=3x$. $\therefore x=8$.

The resultant is therefore a force 3, acting in the direction of the greater force R at a point 8 inches from C in AC produced.

Examples for Exercise.—

1. Two parallel forces, 7 and 10, act in opposite directions at points 6 inches apart. Find a third force which will maintain equilibrium.

2. Parallel forces, 5 and 2, act in opposite directions at a distance of 1 foot. Find their resultant.

Examination on Sections 45—50.

1. What is the condition that two forces, acting on a particle are in equilibrium?

2. Prove that three forces, which are represented in magnitude and direction by the three sides of a triangle taken in order, are in equilibrium.

3. Prove that if the sides of a triangle, taken in order, are parallel to the directions of three forces in equilibrium, they may be taken to represent these forces in magnitude and direction.

4. Enunciate the condition that any number of forces acting on a particle are in equilibrium.

5. Describe an experiment by means of which the parallelogram of forces may be verified.

6. Define a rigid body.

7. What is the condition that two forces, acting on a rigid body, are in equilibrium? Deduce the principle of the transmission of force.

8. Any two parallel forces P and Q act on a body. Find the direction and magnitude of their resultant.

9. Prove that the resultant of two parallel forces acts at a point dividing the distance between their points of application inversely as the forces.

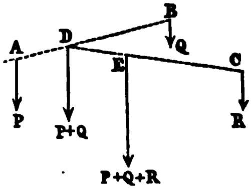
10. Three parallel forces, P, Q, and R, acting on a rigid body, at points A, B, C, are in equilibrium, prove that if R be the greatest force—

(1) $R = P + Q$.

(2) R acts in a direction opposite to that of P and Q.

(3) $P \times AC = Q \times BC$.

51. Resultant of more than Two Parallel Forces.—Let P, Q, and R be three parallel forces acting on a rigid body at the points A, B, and C.



The resultant of P at A, and Q at B has been shown to be equal to P + Q, and to act at a point D in AB, taken so as to satisfy equation (23), which here takes the form—

$$P \times AD = Q \times BD.$$

Similarly the resultant of P + Q at D and R at C is equal to P + Q + R acting at a point E in CD, taken so as to satisfy the equation—

$$(P + Q) \times DE = R \times CE.$$

Thus the resultant of P, Q, and R is a force P + Q + R acting at E.

In the same way, we may prove that the resultant of any number of parallel forces is equal to the sum of the forces, and acts at a point which may be found by the above method. The position of the point depends on the magnitude of the forces, but is independent of the direction in which the forces act.

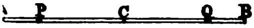
This point is called the *Centre of the Parallel Forces.*

52. Centre of Inertia.—*If a set of parallel forces act on all the particles of a body, and if each force be proportional to the mass of the particle on which it acts, then the centre of these parallel forces is called the Centre of Mass or the Centre of Inertia of the body.*

The attraction of a distant star for the earth or for any terrestial object forms a set of forces, which very approximately fulfils the condition of the above definition.

The earth's attraction for any object also forms such a set of forces, provided that the object be sufficiently small (or sufficiently remote) to allow us to consider the direction of these forces as parallel; with this proviso, the centre of inertia may be properly called the *Centre of Gravity.*

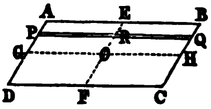
Definition of Centre of Gravity.—*The earth's attraction for any small (or remote) body may be considered to consist of parallel forces acting on the various particles of the body and proportional to their masses. The centre of these parallel forces is called the Centre of Gravity of the body.*



53. Centres of Gravity found geometrically.—(1.) *To find the centre of gravity of a uniform material straight line.*—Let AB be a straight line or thin rod, composed of uniformly heavy matter. Bisect AB in C. Suppose AB to be divided into any number of small equal portions, and let P and Q be two of these portions, and let CP = CQ. Then the weights of the portions P and Q are two equal parallel forces, and therefore (§ 49, Ex. 2) their resultant acts at C.

In the same way we may take the other portions in pairs equally distant from C, and the resultant of the weights of each pair acts at C.

Therefore the resultant of the weights of all the portions acts at C, and C is the centre of gravity of the straight line AB.



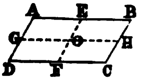
(2.) *To find the centre of gravity of a parallelogram.*—Let ABCD be a parallelogram composed of uniformly heavy matter. Bisect AB in E and draw EF parallel to AD. Suppose the whole parallelogram divided into thin strips by lines drawn parallel to AB, and let PRQ be one of those strips. Then, since AERP and EBQR are parallelograms, PR = AE and RQ = EB.

$$\therefore \text{ PR} = \text{RQ}.$$

$$\therefore \text{ R is the centre of gravity of PQ.}$$

Similarly EF contains the centre of gravity of each strip, and therefore it must contain the centre of gravity of the whole parallelogram.

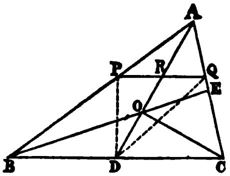
Bisect AD in G and draw GH parallel to AB, meeting EF in O.



Then, as before, by drawing lines parallel to AD, we may divide the parallelogram into thin strips; and the centre of gravity of each strip, and therefore that of the whole parallelogram, lies in GH.

∴ The centre of gravity of the parallelogram is the point O, in which EF and GH intersect.

(3.) *To find the centre of gravity of a triangle.*—Let ABC be a triangle, bisect BC in D, and join AD, and suppose the triangle into thin strips by lines drawn parallel to BC. Let PRQ be one of these strips:



$\because BD = DC,$

$\therefore \triangle BAD = \triangle CAD$ (Euclid i. 38).

Similarly $\triangle BPD = \triangle DQC,$

$\therefore$ the remaining $\triangle APD = \triangle AQD,$

And $PR = RQ.$

For if not, let $PR > RQ,$
Then $\triangle PAR > \triangle RAQ,$*
And $\triangle PDR > \triangle RDQ,$
$\therefore$ by addition $\triangle APD > \triangle AQD.$
But the $\triangle APD$ has been shown to be equal to $\triangle AQD,$
$\therefore PR$ is not $> RQ.$
Similarly PR is not $< RQ.$
And therefore $PR = RQ.$

$\therefore$ R is the centre of gravity of PQ.

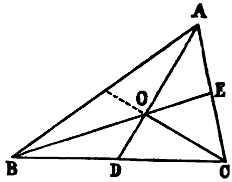
---

* By Euclid i. 35, any parallelogram is equal to the rectangle on the same base and between the same parallels, and therefore its area = base × altitude. Therefore by Euclid i. 41, any triangle = ½ base × altitude. If, therefore, two triangles have equal altitudes, but unequal bases, the greater triangle is that which has the greater base.

In the same way it may be shown that AD contains the centre of gravity of each of the parallel strips, therefore AD must contain the centre of gravity of the whole triangle. Bisect AC in E. Join BE, meeting AD in O. By dividing the triangle into strips parallel to AC, we may prove exactly as above, that BE contains the centre of gravity of the triangle. Therefore the point O, in which AD and BE intersect, is the centre of gravity.

The point O is a point of trisection of each of the lines joining an angular point with the middle point of the opposite side.

Join OC.
$\because AE = EC$,
$\triangle BAE = \triangle BCE$.
Also $\triangle AOE = \triangle EOC$.
$\therefore$ the remaining $\triangle AOB = \triangle BOC$.
Again, $\because BD = DC$.
$\triangle BOD = \triangle DOC$.
$\therefore BOC = 2 \triangle BOD$,
$\therefore \triangle AOB = 2 \triangle BOD$,



and these triangles are between the same parallels.

$\therefore$ The base AO must be equal to twice OD*, or OD is the third part of AD.

Similarly it may be shown that O is a point of trisection of the lines joining B and C with the middle points of the opposite sides.

(4.) *To find the centre of gravity of a circle.*—A circle may be divided into pairs of equal particles, so that each pair consists of two particles on opposite sides of the centre and equally distant from it. Thus, the centre of the circle is the centre of gravity of each pair of particles, and therefore of the whole circle. In the same manner the centre of a sphere may be shown to be its centre of gravity.

(5.) *To find the centre of gravity of a system of bodies whose separate masses and centres of gravity are known.*—Suppose the weight of each body to act at its own centre of gravity, and find the centre of the set of parallel forces thus formed.

---

* As before, this follows from the fact that any triangle $= \frac{1}{2}$ base × altitude. If, therefore, one triangle be double another triangle, and have the same altitude, the base of the first triangle must be double the base of the second.

Example.—Three equal weights are placed at the angular points of a triangle, find the centre of gravity of the system.

Let the weight of each mass be W, and let them be placed at the points A, B, and C, in the last figure. The weights at B and C are equivalent to that of a weight 2W at D, the middle point of B.

We have thus to find the resultant of W at A and 2W at D. We must, therefore, divide AD at O, so that by equation (23)—

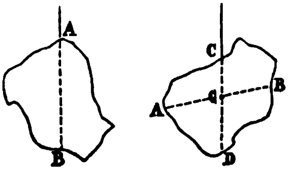
$$W \times AO = 2W \times OD.$$

$$\text{Or } AO = 2\,OD.$$

The centre of gravity of the three weights is therefore the same point as the centre of gravity of the triangle formed by joining the points at which the weights are placed.

Example for Exercise.—Weights of 1, 3, 9, and 3 lbs. are placed at the angles of a square taken in order round the figure, find the centre of gravity of the system.

54. Experimental Method of finding the Centre of Gravity of a Body.—The centre of gravity of a small body may be found experimentally in the following manner. Tie a string to any part of the body and suspend the body by the string. The only forces which act on the body are the tension of the string, acting vertically upwards at the point of suspension, and the weight of the body acting vertically downwards at its centre of gravity. These forces must be equal and opposite, and therefore the centre of gravity must be in the vertical line through the point of suspension.



Now let the body be hung up by any other part. As before, the vertical line through the point of suspension will pass through the centre of gravity. The point where the two lines intersect is therefore the centre of gravity.

In many bodies, as for example a flat plate, the centre of gravity may be readily found by the method described above.

Examination on Sections 51—54.

1. Show how to find the resultant of three parallel forces acting in the same direction.

2. Prove that the resultant of a number of parallel forces is equal to the sum of the forces and acts at a point whose position is independent of the direction of the forces.

3. Explain what is meant by the Centre of Parallel Forces.

4. Define the Centre of Inertia.

5. Define the Centre of Gravity.

6. Find the centre of gravity of a uniform material straight line.

7. Show how to find the centre of gravity of a parallelogram.

8. Show that the centre of gravity of a triangle lies in the line joining any angular point to the middle point of the opposite side.

9. Find the centre of gravity of a uniform flat circular disc.

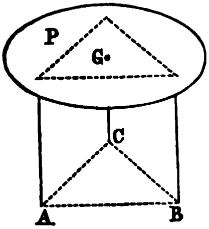
10. Show how to find the centre of mass of a system of bodies whose separate masses and centres of mass are known.

11. Explain a method for finding the centre of gravity of a small body experimentally.

55. Equilibrium of a Body resting on a Surface.—If a body be placed on a surface, the weight of the body causes a pressure on the surface. To this action there is, by the third law of motion, an equal and contrary reaction.

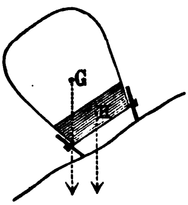
If the body rest on more points than one, there will be a pressure and therefore a reaction at each point on which it rests.

If the resultant of all the reactions be equal and opposite to the weight of the body, the body will be *in equilibrium*.



As an example, take the case of a three-legged table, resting on three legs at the points A, B and C. The resultant of the reactions at A, B and C acts upwards through the centre of gravity G, and is equal to the weight of the body. If, however, a heavy weight be placed on some point, P, outside the triangle formed by the three legs, so that the centre of gravity of the loaded table is removed to some point beyond the triangle, it is clear that the resultant of the reactions at A, B and C can no longer act vertically through the centre of gravity, and the table will consequently fall over.

Thus a body placed on a surface is in equilibrium when the vertical line through its centre of gravity passes through the area formed by joining its points of support.



Example.—A cart loaded with hay is driven along a rough road. Show that it is more likely to upset than an unloaded cart.

Let G be the centre of gravity of the loaded, H that of the unloaded cart; then if, as in the figure, one wheel is considerably higher than the other, the vertical line through G falls outside the base, and equilibrium is impossible for the loaded cart. Under the same circumstances the vertical line through H may, as in the figure, fall within the base, and the unloaded cart may therefore remain in equilibrium.

Example for Exercise.—Sketch a leaning tower and show, by reference to its centre of gravity, that it will fall when made to exceed a certain height.

56. Stable, Unstable, and Neutral Equilibrium.

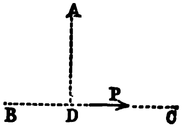
(1.) **Stable Equilibrium.** *A body is said to be in stable equilibrium when, after receiving a small displacement, it tends to return to its former position.* As examples of bodies in stable equilibrium we may mention a weight suspended by a string, and a loaded sphere placed on a flat surface with the loaded part lowest. When either of these bodies receives a small displacement, the centre of gravity is raised, and therefore gravity tends to bring the body back to its former position.

(2.) **Unstable Equilibrium.** *A body is in unstable equilibrium when, after receiving a small displacement, it tends to fall away from its former position.* As examples of unstable equilibrium we may mention a rod balanced on one end, or a loaded sphere with the loaded part upwards. When either of these bodies receives a small displacement, the centre of gravity is lowered, and therefore gravity tends to draw the body away from its first position.

(3.) **Neutral Equilibrium.**—*A body is in neutral equilibrium when, after receiving a small displacement, it remains in the new position.* A sphere on a flat surface is an example of neutral equilibrium. In this case the centre of gravity is neither raised nor lowered by a small displacement.

57. The Moment of a Force about a Point.—*The Moment of a force about a point is the product of the force and the perpendicular on its line of action from the point.*

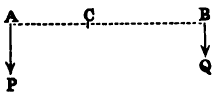
Let P be a force 2 acting along the line BC, and from A let AD be drawn perpendicular to BC, then, if AD = 3 feet, the moment of P about A is P × AD = 2 × 3 = 6.



The moment of P about A represents the power which

P has to turn a body about the point A, if we suppose that point in the body fixed.

It is important to notice that, in the case of parallel forces, the moments of the component forces about any point in the line of the resultant are equal, and tend to turn the body in opposite directions.



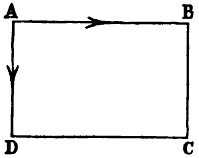
Let P and Q be the two forces, and let the line AB be drawn perpendicular to their direction. Then, if C be a point in the resultant, by (23)—

$$P \times AC = Q \times BC,$$

or the moments of P and Q about C are equal, and it is evident that they tend to turn the body in opposite directions about C.

Examples.—

1. ABCD is a rectangle. Forces represented by AB and AD act at A, prove that their moments about C are equal.



Since ABC is a right angle, the moment of AB about C is AB × BC. It is therefore represented by the area of the rectangle. Similarly, the moment of AD about C is AD × CD, and is also represented by the rectangle. Therefore they are equal.

2. ACB is a straight line, AB = 10 in., and AC = 4 in. If a force of 10 lbs. act at C in a direction perpendicular to ABC, what force must act at B in order that their moments may be equal?

Let P and Q be the forces, then—

$$P \times AB = Q \times AC$$

Substituting their values for AB, AC and Q, this equation becomes—

$$4P = 10 \times 10 = 100.$$

$$\therefore P = \frac{100}{4} = 25.$$

Examples for Exercise.—

1. ABCD is a straight line 10 feet long, and is divided into three equal parts at B and C. Forces of 1, 2, 3 and 4 lbs. act at A, B, C and D perpendicularly to AD. Find their moments about B.

2. Prove that, if two forces act at a point in a body, their moments about any point in their resultant are equal, and tend to turn the body in opposite directions.

Examination on Sections 55—57.

1. What is the condition of equilibrium of a body resting on a surface?

2. What are the three kinds of equilibrium?

3. Define the three kinds of equilibrium, giving examples of each.

4. Define the moment of a force about a point.

5. Prove that the moments of two parallel forces about any point in the line of their resultant are equal.

General Examples in Statics.

1. Three forces, P, Q, and R, acting at a point, are in equilibrium, the angle between P and Q being a right angle—

   (1) Given $P = 15$ and $Q = 8$; find R.
   (2) Given $P = 15$ and $R = 17$; find Q.
   (3) Given the angle between Q and $R = 150^\circ$ and $P = 5$; find R.
   (4) Given the angle between Q and $R = 135^\circ$ and $P = 7$; find Q.

2. Three forces P, Q, and R, acting at a point, are in equilibrium. If $P > Q$, and $Q > R$, prove that the angle between P and Q is the greatest, and that the angle between Q and R is the least.

3. Two equal forces in the same plane act at an angle of 120°, find a third which will maintain equilibrium.

4. Weights of 4 and 6 lbs. are hung at either extremity of a light rod 15 inches long. Find at what point the rod should be supported.

5. P and Q are two parallel forces acting in similar directions

at two points A and B. Their resultant R acts at C. Find R and AC in the following cases:—

(1) P = 7 lbs., Q = 2 lbs., AB = 3 feet
(2) P = 5 lbs., Q = 6 lbs., AB = 1 foot 10 inches.
(3) P = 15 lbs., Q = 5 lbs., AB = 8 inches.

6. P and Q are two parallel forces acting in dissimilar directions at two points A and B. Their resultant R acts at C. Find R and AC in the following cases:—

(1) P = 7 lbs., Q = 12 lbs., AB = 10 inches.
(2) P = 4 lbs., Q = 6 lbs., AB = 1 foot.
(3) P = 15 lbs., Q = 5 lbs., AB = 6 inches.

7. Two men, A and B, carry a weight of 200 lbs. on a pole between them. If the men be 5 feet apart, and the weight be 2 feet from A, what part of the weight will be borne by each ?

8. A well-balanced dog-cart, loaded with 9 cwt., is found on a level road to exert a pressure of 7 lbs. on the horse's back. If the horizontal distance between the pad and the axle be 6 feet, find how far before the axle the centre of gravity of the loaded dog-cart is.

9. Three uniform rods form three sides of a square, find the centre of gravity of the whole.

10. A uniform rod, 2 feet long, and weighing 5 lbs., has a weight of 1 lb. placed at one extremity. Find the centre of gravity of the whole.

11. ABC is a straight line, AB = 5 inches, BC = 3 inches. Weights of 3, 2, and 1 lbs. are laid at A, B, and C respectively. Find the centre of gravity of the system.

12. ABC is a straight line, AB = 7 inches and BC = 9 inches. Weights of 3, 4, and 5 lbs. are placed at A, B, and C. Find the centre of gravity of the system.

13. Find the position of the centre of gravity of a hat, the crown and rim of which are of equal mass and both flat.

14. Prove that the centre of gravity of a parallelogram is the point of bisection of its diagonals.

15. Prove that if the outline of a parallelogram be formed of wire, its centre of gravity is the same as that of the whole figure cut out of paper.

16. Show that the three lines which join the angular points of a triangle to the middle points of the opposite sides, meet in a point.

17. Find by a geometrical construction the centre of gravity of—(1) A quadrilateral; (2) Any rectilineal figure.

18. What is the condition of stability of a given distribution of load on a table?

19. ABC is an equilateral triangle, and AC is bisected in D. Prove that the moment about A of a force represented by BC is twice the moment of a force represented by BD.

20. ABCD is a square. Prove that the moments about A of the forces represented by BC, CD, and BD are equal.

21. Parallel forces P and Q, acting in similar directions at A and B, are balanced by a force R acting at C.

(1) Prove that the moments of P and R about B are equal and opposite.

(2) Prove that the sum of the moments of P and Q about any point in AB produced is equal and opposite to the moment of R about that point.

22. Two forces act at a point. Prove that the algebraical sum of their moments about any point in their plane is equal to the moment of their resultant.

23. Prove that the centre of gravity of a regular polygon is the centre of the inscribed circle.

24. A wire is in the form of a circle on which are strung equal beads separated by equal intervals. Find the centre of gravity of the whole.

25. Determine the character of the equilibrium in the following cases.

(1) A thin book lying on its side.

(2) The same standing on its end.

(3) The same balanced on a corner.

(4) A body of any shape supported at its centre of gravity.

(5) A hemisphere on a horizontal surface with the flat side turned down.

(6) A hemisphere on a horizontal surface with the flat side turned up.

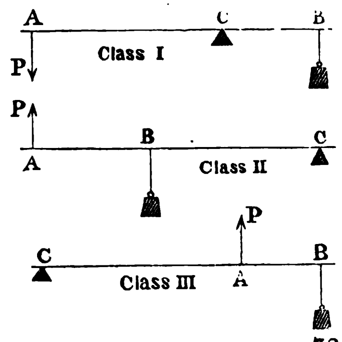
(7) A thin circular disc standing on its edge.

CHAPTER V.

MACHINES.

58. Preliminary.—A machine is an instrument by means of which a force, acting in a given direction at a given point, is made to counterbalance or overcome a resistance, generally acting at a different point and in a different direction. We have already considered cases, in which forces are transmitted by such simple machines as the leg of a table and a stretched cord.

The machines which we proceed to consider in this chapter are (1) the lever and its modifications, namely, the balance, the steelyard, and the wheel and axle; (2) the pulley; (3) the inclined plane. The principles of statics show us under what conditions the applied force is in equilibrium with the resistance. Following the usual custom in elementary treatises on this subject, we call the applied force the Power, and the resistance the Weight, and designate them by the letters P and W.



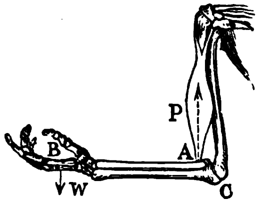
59. The Lever.—*The lever is a rod moveable about a fixed point, which is called the fulcrum.* There are three classes of levers, distinguished in the following manner: AB represents a lever moveable about a fulcrum at C, the power P being applied at A, and the weight W at B. In levers of the first class C is between A and B; in those of the second class B is between A and C; and in those of the third class A is between B and C.

Examples.—

Class I. A poker used for raising coals. Here the bar is the fulcrum, the coals are the weight, and the power is applied by the hand. The common balance.

Class II. A loaded wheel-barrow supported by the handles. The fulcrum in this case is the point where the wheel rests on the ground. A chipping knife, in which one end is fixed to the block by a hinge.

Class III. The human fore-arm supporting a weight. Here the elbow is the fulcrum, the power is applied close to the elbow by the biceps muscle, while the weight is held in the hand. The treadle of a turning lathe.



Other machines are formed of two similar levers connected by a joint, which acts as the double fulcrum.

Examples.—

Class I. Scissors, Pincers.
Class II. Nut Crackers.
Class III. Sugar-tongs.

60. Equilibrium of the Lever.—In every case the Power tends to turn the lever in one direction about the fulcrum, while the Weight tends to turn it in the opposite direction. For equilibrium, therefore, the moments of P and W about C must be equal. If we assume for simplicity that P and W each act at right angles to AB,* the moment of P is

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* If either or both forces are not at right angles to the lever, the moments may be found by drawing perpendiculars from the fulcrum on their lines of action, and the condition of equilibrium is the same, viz.: that the moments are equal.

$P \times AC$, and that of W is $W \times BC$, therefore for equilibrium

$$P \times AC = W \times BC.$$

This equation will be seen to be identical with (23), as P and W are two parallel forces, the resultant of which must act through C. (Compare § 57.)

If we divide both sides of this equation by $W \times AC$, it takes the form

$$\frac{P}{W} = \frac{BC}{AC},$$

which is identical with (24), and shows us that P bears to W the same proportion that BC bears to AC.

Thus P is less than W when AC is greater than BC, P is equal to W when AC is equal to BC, and P is greater than W when AC is less than BC.

When P is less than W, the arrangement is said to be one of *mechanical advantage.*

Examples.—

1. Two bodies, weighing 20 lbs. and 4 lbs., balance at the extremities of a lever 2 feet long. Find the position of the fulcrum.

By (23) $P \times AC = W \times BC$

Here $P = 20$, $W = 4$.

Let $AC = x$ inches, then $BC = 24 - x$ inches.

$$\therefore 20x = 4(24 - x) = 96 - 4x.$$

$$\therefore 24x = 96.$$

$$\therefore \quad x = \frac{96}{24} = 4 \text{ inches.}$$

2. In a lever of the second class, a weight of 3 lbs. is suspended at a point 2 inches from the fulcrum. At what distance must a force of 2 ounces act upwards to produce equilibrium?

By (24) $P \times AC = W \times BC.$

In this case $P = 2$ oz., $W = 48$ oz., $BC = 2$ in.

∴ If AC = x inches, the equation becomes

$$2x = 48 \times 2 = 96.$$

$$\therefore x = 48 \text{ inches or } 4 \text{ feet.}$$

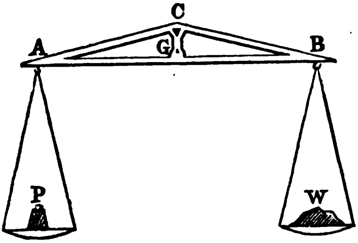
Examples for Exercise.—

1. Weights of 3 lbs. and 9 lbs. balance each other at the extremities of a lever AB. Find the position of the fulcrum C.

2. A straight rod, 6 feet long, projects horizontally from a wall, to which it is attached by a horizontal hinge, and a weight of 10 lbs. is hung from its extremity. What pressure upwards must be exerted at 2 feet from the hinge in order to maintain the rod in its horizontal position?

3. In a lever of the third class, AB is 4 inches and the power is twice the weight. Find AC.

61. The Balance.—The common balance is a lever of the first order, in which AC = BC, and therefore P = W. AB is called the beam. Its centre of gravity G is a short distance vertically below C. The scales which are of equal weight, are hung from A and B. The substance to be weighed is placed in one scale and masses of known weight are placed in the other until the beam rests in a horizontal position.



The qualities of a good balance are *stability* and *sensibility*.

A balance is *stable* if it tends after being disturbed to return quickly to its position of equilibrium. By lengthening CG the stability of a balance is increased, as we thus increase the moments of the forces which tend to bring AB back to the horizontal position.

A balance is *sensible*, if AB deviates perceptibly from its horizontal position when P and W differ by a very small quantity. The sensibility of a balance is increased by lengthening AC and BC.

62. False Balances.—A balance gives false results when its arms are of unequal length. In this case there are two methods by which the true weight of a body may be determined.

The first method consists in placing the body to be weighed in one scale, and filling the other with shot, or other substances, until the beam is horizontal. The body to be weighed is then removed, and known weights are placed in its stead until the beam is again horizontal.

It is clear that the true weight is thus obtained, as the same force must have been exerted in each case in order to maintain equilibrium.

The other method consists in weighing the body in each scale, and calculating its true weight from the results.

Let W be the true weight of the body, P and Q its apparent weights in the two scales. Let a and b be the lengths of the arms.

Then by (23) $\qquad Wa = Pb$

And $\qquad Wb = Qa.$

Multiplying together the corresponding sides of these two equations, we obtain

$$W^2ab = PQ\,ab.$$
$$\therefore W^2 = PQ.$$
$$\therefore W = \sqrt{PQ}.$$

The true weight is therefore found by multiplying together the apparent weights and extracting the square root of this product.

Example.—

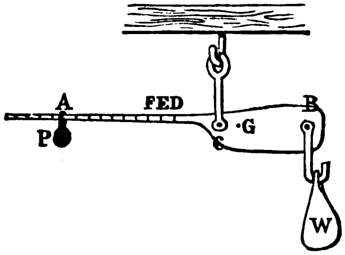
A body weighs 10 lbs. in one scale and $12\frac{1}{10}$ lbs., in the other. Find its true weight.

$$\text{Here } W^2 = PQ = 10 \times 12\tfrac{1}{10} = 121$$
$$\therefore W = 11 \text{ lbs.}$$

Example for Exercise.—

A body weighs 8 lbs. in one scale, and 12 lbs. 8 oz. in the other. Find its true weight.

63. The Steelyard.—The common steelyard is a lever of the first class, and consists of a steel rod suspended from a point C, which is the fulcrum. The body to be weighed W is suspended from a fixed point B, while a moveable weight P is made to slide along the graduated bar AC until there is equilibrium. The position of P, when equilibrium is obtained, indicates the weight of the substance.



AC is graduated in the following manner. First, no weight is hung at B, and P is made to slide along AC until a point D is found, such that the rod is in equilibrium. Then if Q be the weight of the bar and G its centre of gravity

$$P \times CD = Q \times CG.$$

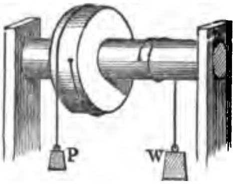
A weight of 1 lb. is now hung at B, and P is moved along the rod until a point E is found, such that the rod is in equilibrium.

The moment of P is now $P \times (CD + DE)$, or $P \times CD + P \times DE$. Now $P \times CD$ is just equal to the moment of the weight of the bar, and therefore $P \times DE$ is equal to the moment of 1 lb. at B. The mark, 1 lb., is therefore placed at E.

Next mark off $EF = DE$. Then $P \times DF = 2P \times DE =$ the moment of two pounds at B.

The mark, 2 lbs., is therefore placed at B, and by marking off equal distances along AC the remainder of the steel-yard is graduated.

64. The Wheel and Axle.—The simplest form of the wheel and axle consists of two cylinders with the same axis. The larger cylinder is called the wheel and the smaller the axle. The weight is hung from the axle by a cord, which is wrapped round its circumference, and the power is applied to the wheel in a similar way; the two cords being wrapped in opposite directions.

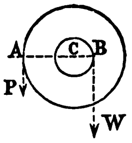


Another form of the wheel and axle is the windlass, in which the power is applied by means of a handle.

A third form is the capstan, in which the axis is vertical, and the power is applied horizontally by means of spokes.

Another common modification is the toothed wheel.

To find the condition of equilibrium of the wheel and axle, we must consider a cross section and suppose both P and W to act in the plane of the paper. Let C be a point in the common axis, and let P act at A, and W at B. Then since there is equilibrium, the moments of P and W about C must be equal, and therefore $P \times AC = W \times BC$.



The same equation will be obtained, at whatever point in the circumference of the wheel P is supposed to act.

It is evident that we may regard the wheel and axle as a lever whose fulcrum is C.

Example.—The radii of a wheel and axle are 4 feet and 6 inches respectively. What force must be applied to raise a weight of 56 lbs?

As above $P \times AC = W \times BC$.

In this case $AC = 48$ inches, $BC = 6$ inches, $W = 56$ lbs.

$$\therefore\ 48P = 6 \times 56.$$

$$\therefore\ P = \frac{336}{48} = 7 \text{ lbs.}$$

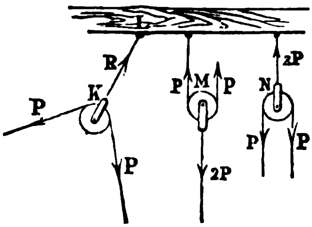
Example for Exercise.—In a capstan the length of the spoke is 6 feet, and the radius of the drum is 1 foot. Find the weight of an anchor which can be raised by 6 men, each of whom exerts a force of 100 lbs.

Examination on Sections 58—64.

1. Define a machine, and explain what is meant by the terms Power and Weight in connection with it.
2. Define a lever. In what three ways may the points of application of the power and weight be situated with regard to the fulcrum? Illustrate by a diagram and give examples of each.
3. What useful machines are formed by joining two similar levers at the fulcrum?
4. Find the condition of equilibrium of the lever.
5. Describe the construction of the common balance. What are the characteristics of a good balance, and how may they be attained?
6. Show how the true weight of a body may be ascertained by means of a balance whose arms are unequal.
7. Describe the steelyard and show how to graduate it.
8. Find the condition of equilibrium of the wheel and axle. Name any modifications of the wheel and axle.

65. The Pulley.—A pulley consists of a small wheel in the rim of which a uniform groove has been cut. It is easily moveable about its axis, which is fixed in a block, the block being either fixed to a beam or suspended by a cord. A cord passes round part of the circumference of the pulley, and is kept stretched by means of forces acting at both ends of the cord. When there is equilibrium these forces must be equal, as otherwise the cord would be drawn round the pulley in the direction of the greater force. Thus if a force P act on one end of a cord which passes round one or more pulleys, an equal force P must act on the other end of the cord. Since this force P keeps the cord stretched, it is called the tension of the cord.

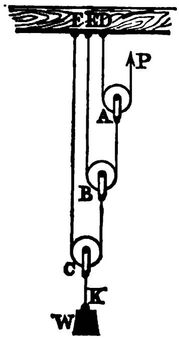
Let K be a pulley supported by a cord KL, and acted on by a second cord whose tension is P. If R be the tension of the cord KL, then the three forces, P, P and R, must be in equilibrium, or R is equal and opposite to the resultant of P and P.



In the pulleys M and N, the forces P and P are parallel, and therefore the resultant is in each case equal to 2P.

There are various combinations of pulleys, and in each we can find the relation of the Power and the Weight, by considering the equilibrium of each pulley separately. The three following cases have been called the first, second, and third systems of pulleys. There is no special reason for this order, but it is retained here for convenience of reference.

66. First System of Pulleys.—In this system a number of cords are used, each of which is attached to the beam and supports one moveable pulley. In the figure the power P acts on the cord AD, which is attached to the beam at D and supports a pulley at A. To this pulley is attached a cord ABE, which supports a second pulley B, and so on.



The last pulley supports the weight W.
Since the tension of AD is P,
∴ the tension of ABE is 2P (§ 65),

$$\therefore \text{ the tension of BCF is } 4P,$$
$$\text{,, \quad ,, \quad CK is } 8P,$$
$$\therefore W = 8P.$$

This may be written $W = 2^3P$. If 4 strings are used, we have in like manner $W = 2^4P$, and so on. Therefore, if n strings are used,—

$$W = 2^nP \quad . \; . \; . \; . \; . \; . \; . \quad (25)$$

The weight W is supported by the force P and by the beam DEF. Therefore the part of the weight supported by the beam is—

$$W - P = 7P.$$

This may also be proved by adding the tensions of the three strings which are attached to the beam.

$$\text{Thus the tension of AD} = P,$$
$$\text{,, \quad ,, \quad BE} = 2P,$$
$$\text{,, \quad ,, \quad CF} = 4P.$$
$$\therefore \text{ Weight supported by beam} = 7P.$$

In the same manner, when there are n strings, the weight supported by the beam is—

$$2^nP - P \text{ or } (2^n - 1)P.$$

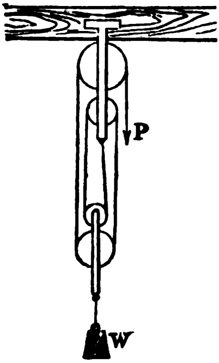
Example—A weight of 1 lb. is sustained by a force of 1 oz. How many cords and pulleys are employed?

$$\text{By (25) } W = 2^nP.$$
$$\text{But } P = 1 \text{ oz., and } W = 16 \text{ oz.}$$
$$\therefore 16 = 2^n.$$
$$\text{But } 16 = 2^4,$$
$$\therefore n = 4,$$

or the system has 4 cords supporting 4 moveable pulleys.

Example for Exercise.—If five pulleys are used, and the Power is 3 lbs., find the Weight.

67. Second System of Pulleys.—Here there are two blocks of pulleys, one of which is fixed to the beam, while the other supports the weight. Only one cord is used, which passes round a pulley in the upper and lower block alternately, and is finally fixed to one of the blocks.



In finding the condition of equilibrium we have only to observe how many times the tension of the string acts on the lower block. In the figure there are 4 strings at the lower block,* and therefore—

$$W = 4P.$$

If there were n strings at the lower block we should have in like manner—

$$W = nP \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In this case the beam supports both W and P. Thus the weight supported by the beam is $W + P = 5P$. This result may also be obtained by observing that there are 5 strings acting on the upper block.

In the same manner, if there are n strings at the lower block, the weight supported by the beam is

$$nP + P \text{ or } (n + 1)P.$$

This system of pulleys is of much practical use. In the other systems too much space is taken up by the motion of the pulleys. In the common form of block all the wheels (or sheaves) are on the same axle.

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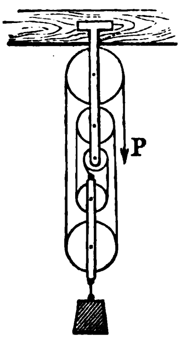
* The number of strings at the lower block is equal to the whole number of pulleys in both blocks.



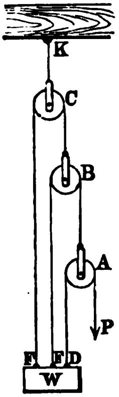
Example.—The upper block contains 3 sheaves, and the string is attached to the lower block. What Weight is sustained by a Power of 1 lb.?

Drawing the figure we can see that there are 5 strings at the lower block, therefore—

$$W = 5P = 5 \text{ lbs.}$$

Example for Exercise.—1 lb. supports 6 lbs. Draw the diagram, and find how many sheaves there are in each block.

68. Third System of Pulleys.—The third system of pulleys is simply the first system inverted, the beam and weight changing places.



Thus the tension of AD is P, that of ABE is 2P, of BCF, 4P, and of CK, 8P.

Thus the beam supports a weight of 8P, and since this is made up of W and P, we have

$$W = 8P - P = 7P.$$

Similarly when these are n strings the beam supports a tension of 2^nP, and therefore $W = 2^nP - P$.

$$\text{Or } W = (2^n - 1)\,P \quad . \quad . \quad (27).$$

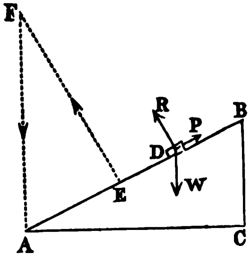
Example.—If there are 4 pulleys and the Power is 3 lbs., find the Weight.

$$\begin{aligned} \text{By (27) } W &= (2^n - 1)\,P. \\ &= (2^4 - 1) \times 3. \\ &= (16 - 1) \times 3. \\ &= 15 \times 3 = 45 \text{ lbs.} \end{aligned}$$

Example for Exercise.—If there are 5 pulleys, and the Weight is 62 lbs., find the Power.

69. The Inclined Plane.—When a body is in contact with a smooth plane, it generally exerts some pressure upon it, and this pressure is at right angles to the surface of the plane. By the third law of motion the plane exerts on the body an equal and opposite pressure which is called the reaction of the plane (§ 55). In the case of a body lying on a horizontal plane, the reaction of the plane is equal to the weight of the body, but when the plane is inclined to the horizon, the direction of the reaction is no longer opposite to the direction of gravity, and a third force is necessary to maintain equilibrium. We shall consider the conditions of equilibrium in two simple cases.

70. Equilibrium on an Inclined Plane when the Power Acts up the Plane.—Let AB be a section of an inclined plane, made by a vertical plane perpendicular to it. Let the plane of the paper represent the vertical plane, and let AC be a horizontal line, and BC be perpendicular to AC. Let D be a body resting on the inclined plane AB. Then D is kept in equilibrium by three forces, its weight W acting downwards, the power P acting in the direction AB, and the reaction R acting at right angles to the plane.



From AB cut off AE = BC. Draw EF at right angles to AB and equal to AC. Join AF.

The △ FAE = △ ABC in every respect (Euclid I., 4), and therefore the angle FAE = the angle ABC.

But the angles ABC and BAC are together equal to a right angle (Euclid I., 32).

∴ FAE and BAC together make up a right angle.

∴ AF is parallel to BC, and the three sides EF, FA, AE, are parallel to the directions of the forces R, W, P.

If then FA be taken to represent W, AE will represent P, and EF will represent R on the same scale.

But FA = AB, AE = BC, and EF = AC.

∴ If AB be taken to represent W in magnitude, BC will represent P, and AC will represent R on the same scale.

Therefore the Power is to the Weight as the height of the plane is to its length.

Examples.—

1. If the height of the plane be to its length as 3 to 5, what Power will support a Weight of 20 lbs?

Suppose that the length is 5 feet and the height 3 feet.

Then 5 feet represents 20 lbs,
∴ 1 " " 4 lbs,
∴ 3 " " 12 lbs, the Power.

2. A railway train, weighing 126 tons, rests on an incline, and is kept from moving downwards by a force of 15 cwt. What is the slope of the incline?

$$\text{Here } \frac{W}{P} = \frac{126 \times 20}{15} = 168. \quad \therefore W = 168 \text{ P}.$$

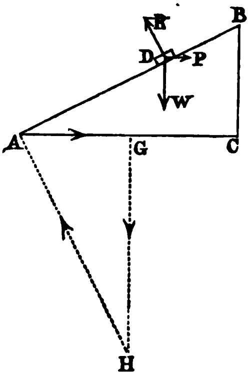
Thus the weight is 168 times the power, and therefore the length of the plane is 168 times its height.

That is, the incline rises 1 in 168.

Examples for Exercise.—

1. If the length of the plane be 13 feet and the base 12 feet, what Weight will be supported by a Power of 5 lbs?

2. The height of a plane is to its base as 12 to 35. If a weight of 3 tons 14 cwt. be placed on the plane, find what force up the plane is required to keep it at rest. Find also the pressure on the plane.



71. Equilibrium on an Inclined Plane when the Power Acts Horizontally.—Let AB represent the inclined plane, D the body resting on it. Let W be the weight of the body, R the reaction of the place, and P the Power acting horizontally.

From AC cut off AG = BC.

From G draw GH at right angles to AC and equal to AC. Join AH.

Then $\triangle$ AGH = $\triangle$ ABC in every respect (Euclid I., 4).

$\therefore \angle$ GAH = $\angle$ ABC.

But $\angle$ ABC + $\angle$ BAC = a rt, $\angle$ (Euclid I., 32).

$\therefore \angle$ GAH + $\angle$ BAC = a rt, $\angle$.

$\therefore$ AH is at right angles to AB, and the three sides AG, GH, HA are parallel to the directions of the forces P, W, and R.

If then GH or AC be taken to represent W in magnitude, AG or BC will, on the same scale, represent P, and AH or AB wlll represent R.

Therefore the Power is to the Weight as the height of the plane is to its base.

Example.—If the height of the plane be to its base as 3 to 8, and the Power be 12 lbs., find the Weight.

Here 3 represents 12 lbs.,
$\therefore$ 1 „ 4 lbs.,
$\therefore$ 8 „ 32 lbs., the Weight.

Example for Exercise.—If the length be to the base as 25 to 24, and the Power be 14 lbs., find the Weight.

Examination on Sections 66—71.

1. What is a pulley? Explain what is meant by the tension of a cord.

2. A rope passes round a pulley, prove that its tension on both sides of the pulley must be equal if equilibrium is maintained.

3. Find the condition of equilibrium in a system of pulleys, in which all the cords are fastened to the beam. Find also the weight supported by the beam.

4. Find the condition of equilibrium in a system of pulleys in which the same cord passes round all the pulleys.

5. Find the condition of equilibrium in a system of pulleys in which all the cords are fastened to the weight.

6. Explain what is meant by the *reaction* of a plane.

7. Find the condition of equilibrium of a body on a smooth inclined plane acted on by a force parallel to the plane.

8. Find the condition of equilibrium of a body resting on a smooth inclined plane, and acted on by a horizontal force.

72. Friction.—When two bodies are pressed together, and we try to make one of them slide over the other, a resistance is experienced which is called *Statical Friction.*

As long as the body is not moved, the amount of friction called into play is just sufficient to maintain equilibrium, and acts in the direction opposite to that in which the forces tend to produce motion.

When the body is on the point of being moved, the *whole amount* of statical friction is called into play.

The whole amount of statical friction between two given bodies bears a fixed proportion to the normal* pressure between the surfaces, if other circumstances remain the same.

Thus if R be the normal pressure, the greatest possible

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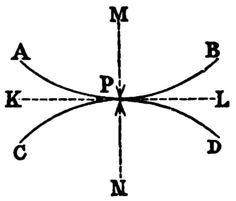
* By *normal pressure* is meant the pressure at right angles to both surfaces. If AB and CD be two surfaces which are pressed together at P, then the portion of their surfaces in contact is approximately a part of a plane surface KL, and the direction of the straight line MPN drawn through P at right angles to KL is the normal to both surfaces.

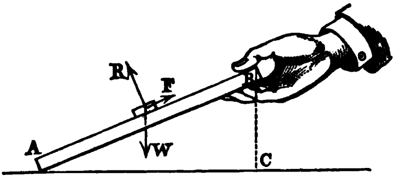


value of the friction is μR, where μ is a fraction depending on the nature of the surfaces, and varying from $\frac{1}{50}$ in such bodies as oiled and polished metals, to $\frac{4}{5}$ in such bodies as wet leather.

μ is called the *co-efficient* of statical friction.

When the statical friction is overcome and motion takes place, a force of friction continues to act, which is proportional to the pressure, and independent of the rate of motion, but the co-efficient of *Kinetic Friction* is less than that of statical friction.

73. **Angle of Repose.**—When a body is laid on a rough plane, and the plane slightly tilted, the force of gravity tends to make the body slide down the plane. This calls into play the force of friction acting up the plane. If W be the weight of the body, R the reaction of the plane, and F the friction, then the three forces W, R, and F are in equilibrium and are therefore (§ 70), proportional to the lengths AB, AC, and BC.



If the plane be now tilted until the body is on the point of sliding, it is easily seen that BC is increased, and therefore F, which is proportional to BC, is also increased.

In the limiting position

$$F = \mu R.$$

but F and R are proportional to BC and AC, and therefore also

$$BC = \mu AC.$$

or dividing both sides of this equation by AC.

$$\mu = \frac{BC}{AC} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

When the plane is in this position the angle BAC is called the *Angle of Repose.*

Examples.—

1. What must the co-efficient of friction be, in order that a body may rest on a plane inclined at an angle of 45° ?

$$\text{By (25)} \quad \mu = \frac{BC}{AC},$$

but if $\angle BAC = 45°$, $AC = BC$,

$$\therefore \mu = 1.$$

This case is imaginary, since μ is less than unity for all known substances.

2. A body just rests on a plane in which the height is one-half the length. Find the angle of repose.

Here ABC is one-half of an equilateral triangle. Therefore the angle of repose is 30°.

$$\text{Also if } BC = 1,\ AC = \sqrt{AB^2 - BC^2} = \sqrt{4-1} = \sqrt{3}.$$

$$\therefore \mu = \frac{BC}{AC} = \frac{1}{\sqrt{3}}$$

Examples for Exercise.—

1. A weight of 80 lbs. rests on a horizontal plane, and it is found that it can be moved by a horizontal force of 10 lbs. Find the co-efficient of friction.

2. A body lies on a horizontal slab 10 inches long. If the co-efficient of friction be $\frac{3}{4}$, how high may one end of the slab be raised, before the body will begin to move ?

74. Machines in Motion.—We have investigated the relation of the Power to the Weight necessary in each case to keep the machine in equilibrium. If the Power be caused to exceed this value by a very small amount the machine will be set in motion, and the force P will do work (§ 27).

In the cases we have considered, no part of the work is spent in giving energy to the machine, and therefore in each case *the work done by the Power is equal to the work done against the Weight.*

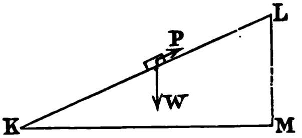
In the cases of the lever and the pulley the points of application of both P and W move in the line in which these forces act, and therefore in these cases, the work done by each force is measured by that force multiplied by the distance moved by its point of application, § 27.

$$\therefore P \times P\text{'s path} = W \times W\text{'s path}.$$

Thus, if P be less than W, its path is greater than that of W in the same proportion, or, as it is sometimes stated, *what is gained in power is lost in space.*

In the case of the inclined plane, in measuring the work done by P and against W, we must consider the distance moved by the body in the directions of these forces respectively.

Thus if the force be parallel to the plane, and the body be moved from K to L the work done by P is P × KL, while the work done on W is W × LM.



Thus the path of the body is increased in the same proportion as the power is diminished, and again we see that what is gained in power is lost in space.

The fact, that the work done by the Power is equal to the work done against the Weight, may be proved in each machine from the conditions of equilibrium. Thus the above equation follows from the result of § 70, which states that P is to W in the same proportion as the height of the plane is to its length, or—

$$\frac{P}{W} = \frac{LM}{KL}$$

If both sides of this equation be multiplied by W × KL, it takes the form—

$$P \times KL = W \times LM.$$

Example.—Show that, in the second system of pulleys, the work done by P is equal to that done against W.

Since there are *n* strings connecting the two blocks, if W be raised 1 foot, each of the *n* strings is shortened by 1 foot, and therefore P descends *n* feet.

But by the conditions of equilibrium,—

$$W = nP,$$

∴ the amount of work done by P in moving *n* feet is equal to that done against W moving 1 foot.

Similarly for any number of feet.

Example for Exercise.—

Prove that the work done by the Power is equal to that done against the Weight in (1) the lever, (2) the first system of pulleys, (3) the wheel and axle.

Examination on Sections 72—74.

1. Define statical friction.

2. When is the whole amount of statical friction called into play, and in what direction does it act?

3. What is meant by the co-efficient of statical friction?

4. What is kinetic friction? In what particulars do its laws resemble those of statical friction, and in what do they differ?

5. Explain what is meant by the angle of repose.

6. Show how we may find the co-efficient of friction, when the angle of repose is given.

7. When a simple machine is in motion, what is the connection between the work done by the power and that done against the weight?

8. Explain by reference to any simple machine, the meaning of the phrase, what is gained in power is lost in space.

General Examples in Machines.

THE LEVER.

1. Where must I place the fulcrum in order that P may

balance W at the extremities of a straight lever AB, in the following cases :—

(1) P = 5 lbs., W = 3 lbs., AB = 3 feet.
(2) P = 6 lbs., W = 4 lbs., AB = 1 foot.
(3) P = 10 lbs., W = ½ lb., AB = 3½ feet.

2. The force at one end of a lever is 3 lbs., and its distance from the fulcrum 2 inches. If the lever be 5 inches long, find the force at the other end.

3. In a pair of nutcrackers the nut is placed one inch from the hinge, and the hand is applied at a distance of 6 inches from the hinge. If the nut require 2¼ lbs. pressure to break it, how much pressure must be exerted by the hand ?

4. If a force of 3 lbs. 10 oz. act on a lever at 1 foot 8 inches from the fulcrum, where must a force of 5 lbs. act to produce equilibrium ?

5. A uniform rod, 2 feet long, weighs 1 lb. ; what weight must be hung at one end, in order that the rod may balance on a point 3 inches from that end ?

6. Two men carry a weight M, by means of a pole AB, one end of which is held by each. The weight is placed at the point C. Find what part of the weight each man bears in the following cases :—

(1) AB = 5 feet, AC = 2 feet, M = 200 lbs.
(2) AB = 7 feet, AC = 1½ feet, M = 350 lbs.
(3) AC = 4 feet, BC = 6 feet, M = 300 lbs.

7. A lever, 5 feet long, rests on a fulcrum at one end, a weight of 10 lbs is suspended 1 foot 10½ inches from the fulcrum, what force acting perpendicularly at the other end will preserve equilibrium ?

8. Two forces, which are as 3 to 2, balance on a lever of the second class. If the distance between their points of application be 2 feet, find the whole length of the lever.

9. Two forces of 5 and 6 lbs. act perpendicularly on a lever at a distance of 5½ inches. Find the position of the fulcrum, when the directions in which the forces act are—(1) similar, (2) dissimilar.

10. A lever 2 feet long is supported by 2 pegs, one below it 3 inches from its extremity, the other above it at its extremity. A weight of 5 lbs. is suspended from the other end. Find the pressure on each peg.

11. A lever, 2 feet long, has the fulcrum at its middle point. Weights of 3, 6, and 12 lbs. are suspended at distances of 4, 8, and 10 inches on one side of the fulcrum. What force must act at the extremity of the other half of the rod to produce equilibrium?

12. Two weights of 4 lbs. are hung on one side of a lever, at distances of 5 and 9 inches from the fulcrum. What vertical force must act at 8 inches from the fulcrum to balance them?

13. A heavy beam, 3 feet long, weighs 4 lbs., and has a weight of 3 lbs. hung at one end. If the fulcrum be 1 foot from the 3 lb. weight, what weight must be hung at the other end to produce equilibrium?

14. A heavy uniform rod has a weight of 3 lbs. hung from one end, and balances about a point 4 feet from that end. If a weight of 5 lbs. be substituted for the 3 lbs. weight, the rod balances about a point 3 feet from the end. Find the length and weight of the rod.

THE BALANCE.

15. A body weighs 4 lbs. in one scale of a false balance and 6¼ lbs. in the other. Find its real weight.

16. The arms of a false balance are 9 and 10 inches long. If the real weight of a body is 5 lbs. 10 oz., how much will it appear to weigh in each scale?

17. A body, which really weighs 3 lbs., appears to weigh 3¼ lbs. in a false balance. Find the proportion of the lengths of the arms.

18. A body, whose real weight is 4 lbs. 8 oz., appears to weigh 4 lbs. in one scale of a balance.

(1) What will it weigh in the other scale?

(2) What is the proportion of the lengths of the arms?

THE WHEEL AND AXLE.

19. The radius of the wheel is 4 feet, and that of the axle 6 inches, what power will sustain a weight of 120 lbs?

20. The radius of the wheel is five times that of the axle. What weight can be raised by a force of 10 lbs.?

21. Twelve sailors, each exerting a force of 96 lbs., work a capstan with levers 7 feet 6 inches long. The radius of the capstan is 16 inches. What weight can they sustain?

PULLEYS.

22. Sketch two systems of pulleys, in each of which the weight is four times the power.

23. Sketch two systems of pulleys in each of which the weight is seven times the power.

24. In the first system of pulleys if there are five pulleys and the weight is 64 lbs., find the power.

25. In the second system of pulleys if the power is 7 lbs. and the weight 56 lbs., draw the diagram.

26. In the first system of pulleys three pulleys are used, each of which weighs 1 lb. Find what weight will be sustained by a power of 10 lbs.

27. In the second system of pulleys, each block contains three sheaves, and the moveable block weighs 3 lbs. What weight will be sustained by a power of 10 lbs.?

28. In the third system of pulleys three pulleys are used, each weighing 1 lb., find the weight if the power be 10 lbs.

THE INCLINED PLANE.

29. On a smooth plane, rising 3 in 5, a weight of 10 lbs. is kept from sliding by a force in the direction of the plane. Find—

(1) this force.

(2) the pressure on the plane.

30. An inclined plane rises 16 in 65. Find what weight will be supported by a horizontal force of 80 lbs.

31. A horizontal force of 35 lbs. supports a weight of 120 lbs. upon an inclined plane. Find the pressure on the plane.

32. A plane is inclined at an angle of 45°. Find the horizontal force necessary to support a weight of 10 lbs.

33. A weight is supported on an inclined plane by a force parallel to the plane. If the power is half the weight, find the inclination of the plane.

34. A weight of 20 lbs. rests on a plane and is supported by a horizontal force of 15 lbs.

(1) Find the pressure on the plane.
(2) Find what power parallel to the plane would support it.
(3) Find the pressure on the plane in this second case.

MISCELLANEOUS.

35. Two levers, each 1 foot long, are combined so that the weight of the first is the power of the second. In the first the power is 9 inches from the fulcrum, and in the second 8 inches. Find the ratio of the power to the weight.

36. Two inclined planes of the same height slope in opposite directions, and two weights rest, one on each plane, connected with each other by a cord passing over a pulley at the common vertex. If the lengths of the planes are 5 feet and 6 feet, find the relation of the weights that equilibrium may be possible.

37. In a lever the weight is four times the power. What will be the velocity of the power when that of the weight is 7 ?

CHAPTER VI.

HYDROSTATICS.

75. Fluids.—It has been already stated that all solid bodies possess a certain amount of rigidity. Liquids and gases have no rigidity. They are considered together under the name of Fluids.

In Hydrostatics we consider only fluids at rest. When *at rest* all fluids satisfy the following definition.

A FLUID *is incapable of resisting a change of shape.*

In other words, fluids do not exert Statical Friction. When *in motion* all known fluids exert Kinetic Friction, and therefore do not satisfy this definition. If a fluid could be found incapable of exerting kinetic friction, it would be called a *Perfect Fluid.*

The different portions of a fluid exert pressure on each other, and on the sides of the vessels which contain them.

In order to investigate fluid pressure we may consider the equilibrium of any small portion of the fluid by itself. The surfaces, by which this portion of the fluid is bounded are pressed by the surfaces (whether solid or fluid) with which they are in contact, and by the third law of motion, they exert on them an equal and opposite pressure.

The pressure exerted by any portion of a fluid at rest is perpendicular to the surface of that portion.

For, if the pressure were not perpendicular to the surface it could be resolved into two components, one perpendicular, and one parallel to the surface. This latter component would tend to produce sliding motion, and since, by the definition, it could not be resisted, motion would take place, and therefore the fluid would not be at rest.

It is assumed throughout this chapter that *no change of temperature* takes place in the bodies treated of.

76. Liquids and Gases.—Fluids are divided into *Liquids* and *Gases*, which differ in the following respects:—

Liquids are nearly incompressible, that is, the volume of a given mass of a liquid remains nearly constant to whatever pressure it is subjected. The volume of a liquid contained in a closed vessel must be either equal to, or less than, the volume of the vessel. In the latter case the liquid only fills part of the vessel, and has, therefore, a free surface. A liquid can also be contained in an open vessel.

Gases are easily compressed, that is, the volume of a given mass of a gas is not constant, but becomes smaller when the pressure to which it is subjected is increased, and larger when the pressure is diminished. (See § 90.) If a quantity of gas is placed in a closed vessel, it expands so as to fill the whole vessel, and then presses on, and is pressed by the sides of the vessel. A gas cannot be contained in an open vessel, unless the space outside the vessel contains some fluid which exerts pressure on the gas. For example, if an open vessel containing a gas be placed in a space exhausted of air, the gas will expand and fill the whole space. On the other hand, if a vessel containing gas be held mouth downwards in water the gas will remain in the vessel.

It is to be observed that we do not in this treatise consider any changes due to such causes as chemical reactions, absorption of gases by liquids, etc.

77. Density.—If we take equal volumes of two bodies, and find that the mass of one is greater than the mass of the other, the body which has the greater mass is said to be *denser* than the other.

Thus, if we compare a cubic foot of iron with a cubic foot of wood, we find that the iron has the greater mass, and is therefore denser than the wood. Similarly mercury is denser than iron, iron than water, water than air.

The DENSITY *of a body is the mass contained in unit volume.* If we take a foot as the unit of length, a *cubic foot* will be the corresponding unit of volume. Let ρ be the density of a body of mass M, whose volume is V cubic feet.

Then the mass of 1 cubic foot is ρ,
,, ,, 2 ,, feet ,, 2ρ,
,, ,, 3 ,, ,, ,, 3ρ,

and so on. Therefore the mass of V cubic feet is $V\rho$, or

$$M = V\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

Since the volume of a given mass of a liquid is constant, its density is also constant, but different liquids have different densities.

In a gas the volume varies with the pressure, and therefore the density also varies. If we divide both sides of (29) by (1) ρ, (2) V, it takes the forms

$$V = \frac{M}{\rho}, \text{ and } \rho = \frac{M}{V},$$

which show us that the volume of a given mass varies inversely as the density.

Examples.—

1. A cubic inch of iron weighs $\frac{1}{4}$ of a lb., find its density.

In the equation $\rho = \frac{M}{V}$, $M = \frac{1}{4}$, $V = \frac{1}{1728}$,

$$\therefore \rho = \tfrac{1}{4} \times 1728 = 432,$$

or the mass of a cubic foot of iron is 432 lbs.

2. A cubic foot of water weighs 1000 ounces, find its density.

Here $M = \frac{1000}{16}$, $V = 1$.

$$\therefore \rho = \frac{1000}{16} = 62\tfrac{1}{2}.$$

Examples for Exercise.—

1. Three cubic feet of oak weigh 219 lbs., find its density.

2. The density of mercury is 812, find the weight of 1 cubic inch.

78. Pressure at a Point in a Fluid.—We have seen that the pressure on the surface of any portion of a fluid at rest is perpendicular to the surface. This pressure may be *uniform*, or may *vary* from point to point of the surface. The pressure at a given point in a given direction is found by considering a plane area including the point and perpendicular to the direction.

If the pressure is uniform over this area, then the pressure at the point is measured by the pressure on a unit of area taken so as to include the point in question. If the pressure is variable, then the pressure at the point is the pressure, which would be exerted on a unit of area, if the pressure all over the unit of area were the same as at the point under consideration.

If we take a foot as the unit of length, a *square foot* will be the unit of area. It is, however, customary in measuring fluid pressure to take an inch as the unit of length and a square inch as the unit of area.

Examples.—

1. An area of 20 square feet sustains a uniform atmospheric pressure, altogether amounting to 43,200 lbs. Find the pressure at a point in the surface.

If a square foot be the unit,—

$$\text{Pressure} = \frac{43200}{20} = 2160 \text{ lbs.}$$

If a square inch be the unit,—

$$\text{Pressure} = \frac{2160}{144} = 15 \text{ lbs.}$$

2. A rectangle measuring 9 inches by 4 undergoes a uniform pressure of 8 lbs. on the square inch; find the pressure on the whole surface.

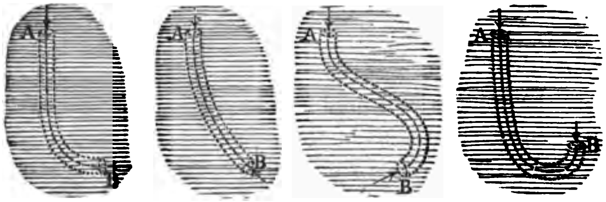
The surface contains 36 square inches,—

$$\therefore \text{Pressure} = 36 \times 8 = 288 \text{ lbs.}$$

Example for Exercise.—If a square area measuring 4 inches each way sustain a uniform pressure of 72 lbs., find the pressure, (1) on a square foot, (2) on a square inch.

79. Fluid not Acted on by External Forces.—*When a fluid is not acted on by external forces, the pressure is the same at all points and in all directions.*

Suppose the fluid to be contained in a closed vessel, and subject only to the pressure of the sides of the vessel.



Let A and B be any two points in the fluid and draw any curved line AB.

Consider the equilibrium of a portion of the fluid in the shape of a circular cylinder of uniform small section described about AB as axis. The cylindrical portion of the fluid is subjected to the pressure of the adjoining fluid on its curved surface and on its ends.

Since the pressures on the curved surface are all at right angles to that surface, they do not tend to prevent the fluid from flowing through the cylinder. Therefore the pressures on the two ends at A and B must be equal, as otherwise the fluid would begin to flow in the direction of the greater pressure. Now the areas of the two ends are equal, and they may be taken as small as we please, therefore the pressures at the points A and B perpendicular to the ends of the cylinder are equal.

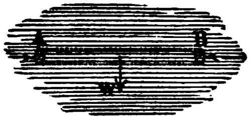
Let us now vary the direction of the line AB, so that the direction of the pressure at A remains the same, while that at B is changed. In each case the pressure at B is equal to the pressure at A, and thus we see that the pressure at B is the same in all directions. And since B is any point in the fluid, it follows that the pressure is the same at all points and in all directions throughout the fluid.

80. Heavy Fluids.—(1) *When external forces, such as gravity, act on a fluid, the pressure at a point is the same in all directions.*

For the preceding demonstration may still be applied by taking the points A and B very near each other, as in that case the weight of the cylindrical portion of the fluid may be neglected when compared with the pressures on its ends.*

(2) *The pressures at any two points in the same horizontal plane are equal to one another.*

Let A and B be two points in the same horizontal plane, and consider a small cylinder of the fluid of which AB is the axis. Since the weight of the fluid acts vertically, it does not prevent motion in either direction. Therefore the pressures at A and B must be equal to each other.



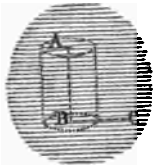
(3) *The pressures at any two points in the fluid differ by the weight of a cylinder of the fluid whose base is unit of area, and whose height is the vertical distance between the points.*

Let A and C be any two points in the fluid. Through C draw a horizontal plane, and through A draw AB perpendicular to this plane.

---

* The pressures on the ends are proportional to their areas, while the weight is proportional to the volume. If the distance AB is reduced to $\frac{1}{10}$ of its former length, the proportions of the cylinder being unaltered, the pressures on the ends are reduced to $\frac{1}{100}$ of their former value and the weight to $\frac{1}{1000}$. Thus the weight may finally be left out of account compared to the pressures.

About AB as axis describe a cylinder whose base is unit of area, and consider the equilibrium of the fluid within the cylinder.



The pressure on the end A and the weight of the fluid tend to make the fluid within the cylinder flow downwards. This is prevented solely by the pressure on the end B upwards, since all the pressures on the curved surface are horizontal.

∴ Pressure on end A + weight of cylinder of fluid = pressure on end B.

But the end A is part of a horizontal plane, and as proved above the pressure on it is uniform, and is therefore equal to the pressure *at the point* A (§ 78);—

∴ Pressure at A + weight of cylinder = pressure at B.

And since C is in the same horizontal plane as B,—

Pressure at C = pressure at B.

∴ Pressure at A + weight of cylinder = pressure at C.

In liquids the density is the same throughout, and hence the weight of the cylinder AB is the same from whatever part of the liquid it is taken, as long as its size is unaltered. In gases, however, it is important to notice that the ends of the cylinder must be in the planes of A and C.

The fact, that the pressure at any point of a fluid is equal to the pressure at its highest point together with the pressure due to the weight of the fluid, is often called the principle of the *Transmissibility of Fluid Pressure.*

Examination on Sections 75—80.

1. Define a fluid. Prove that in a fluid at rest the pressure on any surface is at right angles to that surface.
2. Prove that fluids cannot exert statical friction.
3. What is meant by a perfect fluid?
4. Distinguish between liquids and gases.
5. Define density. Prove the formula $M = V\rho$.

6. Show that the density of a gas is inversely proportional to its volume.

7. Explain what is meant by the pressure of a fluid at a given point in a given direction. How is it measured, (1) when uniform, (2) when variable?

8. Prove that, in a fluid unacted on by external forces, the pressure is the same at all points and in all directions.

9. Prove that the pressure at any point in a heavy fluid is the same in all directions.

10. Prove that in a heavy fluid the pressures at all points in the same horizontal plane are equal.

11. Prove that in a heavy fluid the pressures at any two points differ by the weight of a column of the fluid, whose base is unit of area, and whose height is the perpendicular distance between the points.

12. What is meant by the transmissibility of fluid pressure?

81. Heavy Liquids.—We have seen that in all heavy fluids, and therefore in liquids, the pressures at any two points differ by the weight of a cylinder of the fluid, whose base is unit of area and whose height is the vertical distance between the points.

When a liquid fills a closed vessel, the whole surface of the liquid is generally subjected to pressure by the inner surface of the vessel.

When a liquid has a free surface, the space above this free surface is generally occupied by a gas (as, for example, air or steam), which exerts a nearly uniform pressure on the surface.

Let p be the pressure at the highest point (or at the free surface) of the liquid, and let q be the weight of a cubic foot of the liquid. Then at a depth of 1 foot the pressure is equal to $p + q$, at a depth of 2 feet the pressure is $p + 2q$; and so on.

Therefore at a point x feet below the surface the pressure is $p + xq$.

When liquids are subjected to great pressure as, for example, in Bramah's press (§ 97), the applied pressure is so great that the pressure caused by the weight of the liquid may be left out of account in comparison with it.

That is, in the expression $p + xq$, q is very small compared with p, and we may therefore neglect xq and consider the pressure to be uniform and equal to p.

If, on the other hand, the pressure p at the surface of a liquid is very small, as, for example, in the barometric column (§ 91), we may leave it out of account in comparison with the pressure caused by the weight of the liquid, and the pressure at any point in the liquid is therefore xq. That is to say:—

When there is no pressure at the surface of a heavy liquid, then the pressure at any point is proportional to its depth below the surface.

Examples.—

1. If the pressure of the air on the surface of a lake is 15 lbs. on the square inch, and if a cubic foot of water weighs 1000 oz., find the pressure at a depth of 100 feet.

Take a foot as unit of length. Then $x = 100$.

$\therefore$ The pressure at 100 feet is $p + 100\,q$.

p is the pressure on a square foot at the surface, and $= 144 \times 15 = 2160$ lbs., $q = 1000$ oz. $= 62\frac{1}{2}$ lbs.

$\therefore\ p + 100\,q = 2160 + 6250 = 8410$ lbs.

This is the pressure on a square foot; the pressure on a square inch is found by dividing this result by 144.

2. The pressure at the surface of mercury being zero, find the pressure at a depth of 6 feet, assuming mercury to be $13\frac{1}{2}$ times as dense as water.

Here $p = 0$, and $q = 13\frac{1}{2} \times 62\frac{1}{2} = 843\frac{3}{4}$ lbs. Also $x = 6$.

$\therefore$ The pressure $= 6q = 5062\frac{1}{2}$ lbs.

As before, this is the pressure on a square foot; that on a square inch is $35\frac{5}{32}$ lbs.

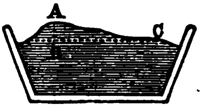
Examples for Exercise.—

1. The pressure at the surface of a liquid is 10 lbs. on the square inch; that at a depth of 4 feet is 20 lbs. on the square inch. Find the weight of a cubic foot of the liquid.

2. If there be no pressure on the surface of water, at what depth will the pressure be 15 lbs. on the square inch?

3. The pressure on the surface of a liquid is 2 lbs. on the square inch, that at a depth of 5 feet is 4½ lbs. Find the pressure at a depth of 400 yards.

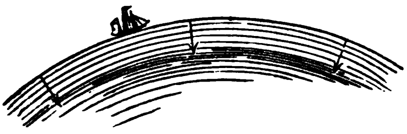
82. **Surface of Heavy Liquids.**—We have seen that in heavy fluids, the pressures at all points in the same horizontal plane are equal. It follows that *the surface of a heavy liquid is horizontal* if all parts of that surface are subjected to the same external pressure.



For, if possible, let A and C be two points in the surface of a heavy liquid, which are subjected to the same external pressure, but are not on the same horizontal plane, and let AB be the perpendicular from A on the horizontal plane through C. Then by § 80 the pressure at C exceeds that at A by the weight of a cylinder of the liquid whose base is unit of area and whose height is AB. Therefore the pressures at A and C cannot be the same unless these points are in the same horizontal plane.

The proposition as above enunciated is only true in the case of small surfaces. In the case of such large areas as lakes, canals, and seas, the directions in which gravity acts at distant points in the surface are not even approximately parallel. Each small part of the surface is very approximately a plane perpendicular to the direction of gravity at that part, but the whole surface is approximately spherical.

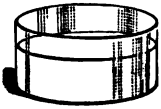
The figure represents a section of the surface of one of the larger oceans made by a plane passing through the earth's centre, the curvature being enormously exaggerated. The arrow heads show the direction of gravity at different points of the surface.



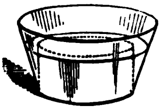
83. Pressures on the Sides and Base of a Vessel containing a Heavy Liquid.—We have seen that the pressure at any point of a heavy liquid is equal to $p + xq$. p represents the atmospheric (or other) pressure on the surface of the liquid, and may be left out of account, in considering the pressure due to the liquid.

Thus the pressure on the interior surface of the vessel at any point is xq, where x is the depth of that point below the surface of the liquid. Suppose the vessels represented in the diagrams below to be filled to the same depth with water, then the pressures at any point in their bases (supposed horizontal) are the same. Thus the whole pressure sustained by the base of each vessel is equal to the weight of a cylinder of water standing on that base and terminated by the surface of the liquid.

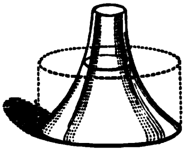
In figure 1, the vessel is cylindrical and the pressure on the base is equal to the weight of all the liquid in the vessel, pressure on the sides being horizontal.



In figure 2, the pressure on the base is less than the weight of the liquid in the vessel, by the weight of the portion of liquid contained in the wedge-shaped ring which surrounds the dotted cylinder. The pressure on the sides may in this case be resolved into two components, one

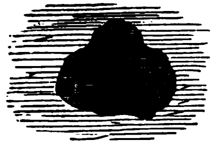


horizontal and the other vertically downwards. The latter is equal to the weight of the liquid in the wedge-shaped ring.



In figure 3, the pressure on the base is more than the weight of all the liquid in the vessel, and the pressure on the sides has a component acting vertically upwards. This component is equal to the weight of the liquid which would be vertically over the sides if the liquid stood at the same level outside and inside the vessel. Thus the resultant downward pressure caused by the liquid is found by subtracting the upward pressure on the sides from the downward pressure on the base, and is, as in the other cases, equal to the weight of the fluid in the vessel.

84. Pressure on Body immersed in a Fluid.—Let S be a solid body immersed in a fluid. Suppose S removed, and the space occupied by it filled by the fluid.



Then the portion of the fluid occupying the space S is acted on only by its weight, and by the pressure of the surrounding fluid, and, since there is equilibrium, these must be equal.

But the pressures of the surrounding portions of the fluid depend only on the depth below the surface, and are therefore the same, whether the space S is occupied by a liquid or a solid.

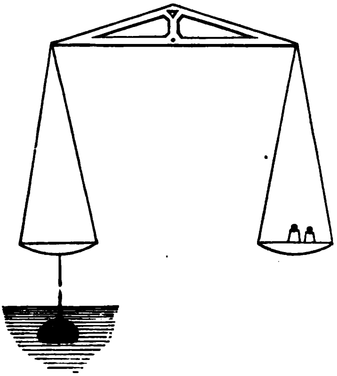
Therefore the resultant pressure on the solid is vertically upwards, and is equal to the weight of that part of the fluid which would fill the same space.

85. Weighing in Air and in Water.—When a body is immersed in any fluid, we have seen that the resultant pressure of the fluid on the body is upwards, and equal to the weight of that portion of the fluid which the body displaces.

Thus when any body is surrounded by air, the air exerts an upward pressure on it equal to the weight of the same volume of air. If, as in the case of a balloon or a mass of heated air, the density of the body is less than that of the surrounding atmosphere, the upward pressure is greater than the weight of the body, and therefore causes the body to ascend. Similarly the downward pressure which any body surrounded by air exerts on the hand, the ground, or the scale of a balance is not, strictly speaking, equal to the weight of the body, but to the weight of the body diminished by the weight of the air which it displaces.

To find the exact weight of a body, it must therefore be weighed in a vacuum. In the case of ordinary solids, however, the loss of apparent weight due to atmospheric pressure is so small that it may be neglected when compared with the whole weight of the body.

On the other hand, when a solid body is immersed in water, oil, or any other heavy liquid, the apparent loss of weight is generally considerable, and we shall presently see that this process, combined with ordinary weighing, enables us to ascertain the density of the body.



The diagram shows how this loss of weight may be ascertained. Bodies soluble in water may be weighed in oil, or they may be encased in a thin film of wax.

Examination on Sections 81—85.

1. Prove that the pressure at any point in a liquid is equal to the pressure on the free surface together with the pressure caused by the weight of the liquid.

2. Explain under what circumstances the pressure in a liquid may be considered to be uniform, and under what circumstances it may be considered to be proportional to the depth below the surface.

3. Prove that the surface of a liquid at rest is a horizontal plane.

4. Explain why the proof of the preceding question applies only to small surfaces.

5. Prove that the pressure on any portion of the horizontal base of a vessel containing liquid is equal to the weight of the liquid, which would be contained in a cylindrical figure erected on that base and terminated by the surface of the liquid.

6. Mention, and illustrate by a diagram, cases of vessels in which the whole pressure on the base is (1) greater, and (2) less than the weight of all the fluid in the vessel.

7. Show that in both cases of the last question the resultant downward pressure on the base and sides of the vessel is equal to the weight of the fluid contained in the vessel.

8. A body is immersed in a fluid. Find the resultant of all the pressures exerted by the fluid on the surface of the body.

9. Prove that, when a body surrounded by any fluid is weighed, the result is not the true weight of the body.

10. Show by examples that fluids exert an upward pressure on bodies immersed in them.

86. Specific Gravity.—The weight of a given volume of any substance compared with the weight of an equal volume of a standard substance is called the *Specific Gravity* of the substance.

The standard substance employed is distilled water at the temperature of its maximum density (4° centigrade).

The specific gravity of water is therefore unity.

It follows that *the* SPECIFIC GRAVITY *of any body is equal* *to the weight of the body divided by the weight of an equal volume of water.*

Let W be the weight of the body, W' the weight of an equal volume of water, then if s be the specific gravity of the body,—

$$\frac{W}{W'} = s; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

or, multiplying both sides by W',—

$$W = sW'.$$

Since the weights of bodies are proportional to their masses,* and the masses of equal volumes of different bodies are proportional to their densities (see § 77), it follows that the specific gravities of bodies are proportional to their densities.

Thus a table of densities may be converted into a table of specific gravities by dividing the density of each substance by that of water.

Examples.—

1. The densities of water, oak, iron, and mercury, were found in § 77 to be 62½, 73, 432, and 812 respectively, find their specific gravities.

Dividing the density of each by that of water we obtain—

$$\text{Specific gravity of water} = \frac{62\frac{1}{2}}{62\frac{1}{2}} = 1.$$

$$\text{,, \quad ,, \quad ,, \quad oak} = \frac{73}{62\frac{1}{2}} = 1{\cdot}2.$$

$$\text{,, \quad ,, \quad ,, \quad iron} = \frac{432}{62\frac{1}{2}} = 6{\cdot}9.$$

$$\text{,, \quad ,, \quad mercury} = \frac{812}{62\frac{1}{2}} = 13.$$

---

* See the law of universal gravitation, § 37. This fact was ennunciated by Newton after a series of experiments in which he investigated the action of gravity by means of a pendulum with a hollow bob into which he introduced various substances.

2. The specific gravity of silver is $10\frac{1}{2}$, find the weight of a cubic inch.

The weight of a cubic inch of water is $\frac{62\frac{1}{2}}{1728}$ lbs.

Thus in the equation $W = sW'$, we have—

$$s = 10\tfrac{1}{2},\ W' = \frac{62\frac{1}{2}}{1728},$$

$$\therefore\ W = 10\tfrac{1}{2} \times \frac{62\frac{1}{2}}{1728} = \frac{875}{2304} \text{ lbs.}$$

3. A piece of wood weighs 23 lbs., and the same volume of water weighs 25 lbs., find the specific gravity of the wood.

By (30) $s = \frac{W}{W'} = \frac{23}{25}$.

4. Thirty-one cubic centimetres of gold weigh 599 grammes, find the specific gravity of gold.

In using the metrical system it is important to remember that a cubic centimetre of distilled water weighs 1 *gramme.*

∴ Thirty-one cubic centimetres of water weigh 31 grammes.

∴ By (30) $s = \frac{W}{W'} = \frac{599}{31} = 19{\cdot}3.$

Examples for Exercise.—

1. A cubic foot of copper weighs 540 lbs.; find its specific gravity.

2. The specific gravity of ice is $\frac{9}{10}$. Find (1) the volume occupied by one ton of ice; (2) the weight of a cubic yard of ice.

3. Sixteen cubic centimetres of zinc weigh 108·8 grammes; find its specific gravity.

87. Specific Gravity found by Weighing in Water.— When the volume and weight of a body are known, its specific gravity may be found by the methods of § 86. When the volume is not known, we may find the weight of an equal volume of water, by weighing the substance in water, and observing by how much the weight in water is less than the true weight.

Thus if W be the weight of the body, w be its weight when weighed in water, $W-w$ is the portion of its weight sustained by the pressure of the water, and is therefore equal to the weight of an equal volume of water (§ 84).

$$\therefore s = \frac{W}{W'} = \frac{W}{W-w} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

Examples.—

1. A piece of ivory weighs 3 lbs. in air and 1·4 lbs. in water, find its specific gravity.

By (31) $s = \frac{W}{W-w} = \frac{3}{3-1\cdot 4} = \frac{3}{1\cdot 6} = 1\frac{7}{8}$.

2. A piece of wood weighing 5 lbs. is attached to a piece of metal weighing 10 lbs., and the whole weighs 7 lbs. in water. The metal alone weighs 9 lbs in water. Find the specific gravity of the metal and of the wood.

For the metal we have by (31)

$$s = \frac{W}{W-w} = \frac{10}{10-9} = 10.$$

To find the specific gravity of the wood, we observe that the whole weight of the body and metal is 15 lbs., and therefore the upward pressure of the water sustains a weight of 8 lbs. But the water displaced by the metal weighs 1 lb. Therefore the water displaced by the wood weighs 7 lbs.

$\therefore$ By (30) $s = \frac{W}{W'} = \frac{5}{7}$.

Examples for Exercise.—

1. A certain substance weighs 14 lbs. in air and 10 lbs. in water. Find its specific gravity.

2. A body A, weighing 10 lbs., is attached to a body B, which weighs 6 lbs., and the whole is found to weigh 3 lbs. in water. If A weigh 5 lbs. in water, find the specific gravity of each.

3. A piece of cork, weighing 1 lb., is attached to 21 lbs. of silver, and the whole weighs 16 lbs. in water. If the specific gravity of cork is $\frac{1}{4}$, find that of silver.

88. Specific Gravity of Fluids.—The specific gravity of any fluid may be ascertained by weighing a given body both in water and in the fluid, and comparing the apparent losses of weight.

Thus if W be the real weight of the body, w its weight in water, and w_1 its weight in the fluid, $W - w$ is the weight of the water displaced by the body, and $W - w_1$ is the weight of the fluid displaced.

Thus the weight of a fixed volume of the fluid, divided by the weight of an equal volume of water is—

$$s = \frac{W - w_1}{W - w} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

A more direct method of finding the specific gravity of a fluid is shown below in example 4.

Examples.—

1. A body weighs 1000 grains in air, 300 grains in water, and 420 grains in another fluid, find the specific gravity of the latter fluid.

By (32) $$s = \frac{W - w_1}{W - w} = \frac{1000 - 420}{1000 - 300} = \frac{580}{700} = \frac{29}{35}$$

2. A solid, weighing 10 lbs., weighs 6 lbs. in a fluid A, and 8 lbs. in a fluid B, compare the specific gravities of A and B.

Here A supports 4 lbs., B 2 lbs. Thus the weights of equal volumes of A and B are in the proportion of 4 to 2, that is 2 to 1.

Therefore the specific gravity of A is twice that of B.

3. A body weighs 5120 grains in a vacuum, 4120 in distilled water, 4095 in salt water, find the specific gravities of the body, and of the salt water.

By (31) the specific gravity of the body is

$$s = \frac{W}{W - w} = \frac{5120}{5120 - 4120} = 5{\cdot}12.$$

By (32) the specific gravity of the salt water is—

$$s = \frac{W - w_1}{W - w} = \frac{5120 - 4095}{5120 - 4120} = 1 \cdot 025.$$

4. A bottle weighs 3 oz. It is filled with water and found to weigh 12 oz.; it is then filled with oil, and found to weigh 11 oz. Find the specific gravity of the oil.

Here the weights of equal volumes of the water and oil are found, by subtracting the weight of the bottle, to be 9 oz. and 8 oz. respectively.

$$\therefore \text{ By (30) } s = \frac{W}{W'} = \frac{8}{9}.$$

5. Three lbs. of a fluid A, whose specific gravity is $1\frac{1}{4}$, are mixed with 2 lbs. of a fluid B, whose specific gravity is $\frac{3}{4}$. Find the specific gravity of the mixture.

Here the whole weight is 5 lbs., and we must find the weight of the water which would have the same volume.

In fluid A, $\frac{5}{4}$ lb. occupies the space of 1 lb. of water,
∴ 1 lb. „ „ $\frac{4}{5}$ lb. „
∴ 3 lbs. „ „ $2\frac{2}{5}$ lbs. „
In fluid B, $\frac{3}{4}$ lb. „ „ 1 lb. „
∴ 1 lb. „ „ $\frac{4}{3}$ lb. „
∴ 2 lbs. „ „ $2\frac{2}{3}$ lbs. „

∴ 3 lbs. of A + 2 lbs. of B occupy the space of $2\frac{2}{5} + 2\frac{2}{3} = 5\frac{1}{15}$ lbs. of water.

$$\therefore \text{ By (30) specific gravity } = \frac{5}{5\frac{1}{15}} = \frac{75}{76}.$$

Example for Exercise.—

1. A body weighs 5 oz. in a vacuum, 4 in water, and 3 in another liquid. Find the specific gravities of the body and the second liquid.

2. A body weighing 4 lbs. is found to weigh 1 lb. in a liquid whose specific gravity is $\frac{1}{2}$. Find the specific gravity of the body.

3. A bottle weighs 10 grains when exhausted of air. When

air is admitted it weighs 10¼ grains. When filled with water it weighs 510 grains. Find the specific gravity of air.

4. Find the specific gravity of a mixture containing 4 cubic inches of copper, specific gravity 9, and 6 cubic inches of zinc, specific gravity 7, assuming that the volume of the mixture is 10 cubic inches.

89. Floating Bodies.—When a body is specifically lighter than a liquid in which it is immersed, the upward pressure of the liquid (§ 84) causes the body to rise to the surface, and to float partly above and partly below the surface.

When in equilibrium, the weight of the body is supported by the upward pressure of the liquid which is equal to the weight of the fluid displaced. Thus the weight of the fluid displaced is equal to the weight of the floating body.

If, then, the body is of uniform density, and we know what fractional part of the body is immersed, we can find its specific gravity.

Let W be the weight of the body, and let $\frac{x}{y}$ of the body be immersed. Then, the weight of the portion immersed is $\frac{x}{y}$ of the weight of the whole body, and is therefore equal to $\frac{xW}{y}$. Dividing the weight of this portion by W, the weight of the water which it displaces, we obtain.

$$s = \frac{x}{y} \quad . \quad . \quad . \quad . \quad (33).$$

When this fraction cannot be ascertained, the specific gravity may be found by attaching a sinker, and proceeding in the manner shown in § 87, Example 2.

Examples.—

1. A body floats with three-fourths of its volume immersed. Find its specific gravity.

$$\text{By (33)} \quad s = \frac{x}{y} = \frac{3}{4}.$$

2. A rod of wood floats with one-third of its volume out of the water. Find its specific gravity.

$$\text{Here the portion immersed} = 1 - \frac{1}{3} = \frac{2}{3} \text{ of the rod.}$$

$$\therefore \ s = \frac{2}{3}$$

3. The specific gravity of mercury is $13\frac{1}{2}$. A piece of iron containing 27 cubic inches floats in mercury with 15 cubic inches below the surface. Find its specific gravity.

If the specific gravity of mercury were taken as unity, that of iron would be $\frac{15}{27}$, but since the specific gravity of mercury is $13\frac{1}{2}$, that of iron is $\frac{15}{27}$ of $13\frac{1}{2} = 7\frac{1}{2}$.

Examples for Exercise.—

1. What is the specific gravity of a piece of iron which floats with $\frac{3}{40}$ of its bulk above the surface of water?

2. What is the specific gravity of a body which, when placed in water, is wholly submerged, but does not sink?

3. A body, which weighs 6 lbs., floats in a liquid with one-third above the surface. What pressure must be applied to keep the body wholly submerged?

Examination on Sections 86—89.

1. Define Specific Gravity. How is it measured?

2. What substance is taken as a standard in determining specific gravities?

3. Show that the specific gravities of any number of substances are proportional to their densities, and explain how the numerical values of their specific gravities may be obtained from the numerical values of their densities.

4. Explain how the specific gravity of a substance may be approximately obtained by weighing it in air and in water, and

prove that, if W be the weight of the body, and w its weight in water, the specific gravity is equal to $\frac{W}{W-w}$.

5. Explain two methods by which the specific gravity of a fluid can be ascertained.

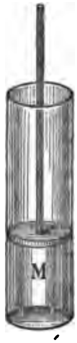
6. Prove that the specific gravity of a floating body bears to that of water the same proportion that the volume of the part immersed bears to the volume of the whole.

7. Mention any two methods of finding the specific gravity of a body which floats in water.

90. Boyle's Law.—*This law, which connects the pressure and volume of gases,* was discovered by the Honourable Robert Boyle, one of the founders of the Royal Society of London. It is sometimes called *Mariotte's Law,* after a French physician Mariotte, who is believed to have discovered it independently; but there can be no doubt as to Boyle's priority, as his work on gases was published in 1662, while Mariotte's was published fifteen or eighteen years later.

It is as follows:—

The pressure of a gas is inversely proportional to the volume.



Let M be a quantity of gas enclosed in a cylinder, occupying a volume V, and subjected to a pressure p by means of a piston. By § 81 the pressure at any point in the gas is equal to p + the pressure due to the weight of the gas. But in this case the pressure due to the weight is so small, that we may neglect it, and consider the pressure at any point in the cylinder to be equal to p. If now the pressure on the piston be increased to $2p$, the piston will be forced down until the volume of M is reduced to $\frac{1}{2}$V, when the pressure at every point throughout the gas will be equal to $2p$. Similarly, if the pressure be made $3p$, the volume will be reduced to $\frac{1}{3}$V. If, on the other hand, the pressure on the cylinder be diminished to $\frac{1}{2}p$, the gas will expand, forcing up the piston until the volume is 2V. If the pressure be diminished to $\frac{1}{3}p$, the volume will increase to 3V, and so on.

This may be expressed by the equation,—

$$p = \frac{C}{V} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

which may also be written, $pV = C$ or $V = \frac{C}{p}$, C being constant and equal to the pressure of the gas at unit volume. Comparing (34) with the equation (§ 78)—

$$\rho = \frac{M}{V},$$

we see that Boyle's Law may be expressed in this form :—

The pressure is proportional to the density.

Examples.—

1. A bladder contains 24 cubic inches of air when exposed to the pressure of the atmosphere. Find its volume when (1) the pressure is increased to 3 atmospheres ; (2) diminished to $\frac{1}{4}$ of an atmosphere.

(1.) Since the pressure is tripled, the volume will be reduced to $\frac{1}{3}$, and is therefore $\frac{1}{3}$ of $24 = 8$ cubic inches.

(2.) Since the pressure is diminished to $\frac{1}{4}$, the volume will be increased to four times its former value, and is therefore $4 \times 24 = 96$ cubic inches.

2. The pressure on the surface of the sea is 2160 lbs. on the square foot, and the weight of a cubic foot of water is $62\frac{1}{2}$ lbs. If a diving bell contain 179 cubic feet of air at the surface, into what space will the air be compressed when the bell is lowered to a depth of 80 feet?

Using the formula $p = \frac{C}{V}$, at the surface of the sea,—

$$p = 2160 \text{ and } V = 179.$$

$$\therefore C = pV = 2160 \times 179.$$

By § 81 the pressure at a depth of 80 feet is—

$$p + xq = 2160 + 80 \times 62\tfrac{1}{2} = 7160.$$

Call this pressure p' and the volume of the air in the diving-bell at this depth V′. Then by (34)—

$$V' = \frac{C}{p'} = \frac{2160 \times 179}{7160} = \frac{54 \times 40 \times 179}{40 \times 179} = 54.$$

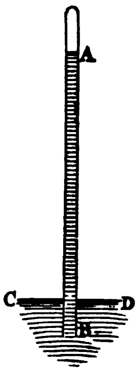
Examples for Exercise.—

1. A cubic foot of air weighs 570 grains at a pressure of 15 lbs. on the square inch. What will a cubic foot of air weigh at a pressure of 16 lbs. on the square inch?

2. If the specific gravity of air at the surface of the sea be ·001, at what depth will the air in a bladder be compressed to the density of water, assuming the pressure at a depth of 100 feet to be four times the pressure of the atmosphere?

91. The Barometer.—The pressure of the atmosphere is ascertained by the barometer. This instrument is constructed in the following manner:—

A glass tube AB, about 32 inches long, sealed at one end A, is filled with mercury. The end B is then closed and placed under the surface of some mercury in a vessel, and the tube is held in a vertical position. When B is opened the mercury in the tube falls a short distance, leaving an empty space above, but the surface remains about 29 or 30 inches higher than that of the mercury in the vessel.

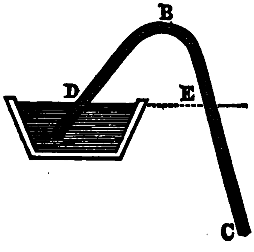


The space above the mercury in the barometer contains a little vapour of mercury, but is practically a perfect vacuum. There is therefore no pressure on the surface of the mercury within the tube, and the pressure at any point below is proportional to its depth below this surface. The pressure at the level CD is equal to the pressure of the atmosphere. Thus the pressure of the atmosphere is proportional to the weight of the mercury in the tube of the barometer. A barometer may be constructed of any liquid, but mercury is nearly always employed on account of its great density. If water be employed, since mercury is 13½ times as dense as water, the column of water will be 13½ times the length of the column of mercury, that is, about 33 feet. At any point above the surface of the earth, the pressure is, by § 80, less than that on the surface of the earth, and therefore the mercury in the barometer is lower. Thus the height of a mountain, or of the ascent of a balloon, may be approximately found by means of a barometer.

h/

Since by Boyle's law the density is proportional to the pressure, the atmosphere becomes less dense the farther it is removed from the surface of the earth, thus differing altogether from the water of the sea, which retains very nearly the same density at all depths, although the pressure at the lowest depths is very great indeed.

92. The Syphon.—The syphon is a bent tube open at both ends. It is used to bring a liquid from a higher to a lower level over an obstacle such as the side of a vessel.



Let ABC be a bent tube, and let it be filled with the same liquid as that which is contained in the vessel to be emptied. Let both ends be closed, and one end A placed below the surface of the liquid in the vessel, the end C being outside the vessel and also below the surface.

Let the end A be now opened, and let D and E be the points where the syphon meets the plane of the surface of the liquid. Then, since the pressure is the same at all points in the same horizontal plane, the pressures within the syphon at D and E are each equal to the pressure of the atmosphere on the surface of the liquid in the vessel. This pressure will support the portion of the liquid DBE, unless the height of B above the surface is greater than the height of a barometric column of the liquid.

At the end C the pressure of the liquid within the syphon is, by § 81, greater than the pressure at E, and therefore greater than the pressure of the atmosphere.

It follows that when C is opened, the pressure of the atmosphere will be unable to sustain the liquid in the tube, and it will therefore flow through the tube until the level of the liquid in the vessel falls below either A or C.

93. The Air Pump.—This is a machine for diminishing the quantity of air in a vessel. It consists of a cylinder AB, in which a piston P is moved alternately up and down a tube leading from the cylinder to the vessel V from which air is to be removed. In the piston P, and at the bottom of the cylinder B, there are valves opening upwards, that is, apertures which are closed, if the pressure of the air above them is greater than that below, but which immediately open, and allow air to pass through them, if the pressure below is greater than the pressure above.

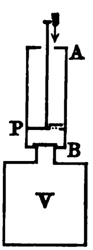


Figure 1.

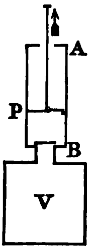


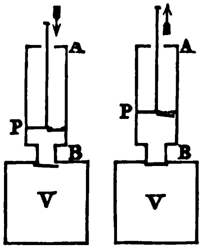
Figure 2.

When the piston is pushed down, as in figure 1, the air in PB is condensed, and therefore, by Boyle's Law, its pressure is increased. The effect of this is to close the valve at B, and open the valve at P. As the piston descends, the air escapes through P, and finally a very small quantity of air remains in the cylinder. The piston is now drawn up, which causes the air in PB to expand, and therefore to exert less pressure than the outside atmosphere. The valve P is therefore closed and that at B opened.

As the piston rises the air in V expands, and fills both the vessel and the cylinder. When the motion of the piston is reversed, the valve B is immediately closed, and as soon as the density of the air in PB exceeds that of the outer atmosphere, the valve P is opened (as in figure 1), and the process continues as before. In this way more and more air can be removed from V, but, as only a part of what remains can be removed by each stroke, we can never, by this process, obtain an absolute vacuum.

In Sprengel's air pump the vessel from which air is to be taken is placed in communication with the vacuum of a barometric tube, and the air is removed by a continuous flow of mercury from a vessel at the top of the tube. By its use the density of air may be reduced to $\frac{1}{1000000}$ of its ordinary value.

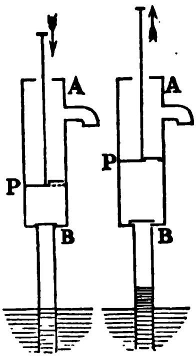
94. The Condenser.—If both valves of an air pump are constructed to open downwards instead of upwards, it becomes a condenser, an instrument for forcing additional air into a vessel. When the piston is lowered (as in figure 1), the air in PB is condensed, the valve P is closed, and as soon as the density of the air in PB exceeds that of the air in V, the valve B is opened and air enters V.



When the piston is raised (figure 2), the air in PB expands, and its pressure being diminished, the valve at B is closed, and as soon as the density of PB is less than that of the atmosphere, the valve P is opened and air enters the cylinder. The alternate motion of the piston is continued, and more air is forced into P with each stroke.

The efficiency of both the air pump and the condenser is increased by making the piston rod work in an air-tight collar and adding a valve at A, as in that case the density of the air in PB need only exceed the density of that in AP in order to open (or close) the valve at P.

95. The Common Pump.—This pump (often called the *Suction Pump* or the *Lifting Pump*), is used for raising water and other liquids. Its construction is the same as that of the air pump, except that the tube at the bottom of the cylinder, instead of leading into a closed vessel, dips into the liquid to be raised. The working of the pump removes air from this tube. The pressure of the atmosphere on the surface of the liquid outside causes the liquid in the tube to rise, until the weight of the column of liquid in the tube, together with the diminished pressure of air, are equal to the atmospheric pressure.

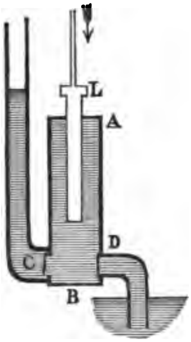


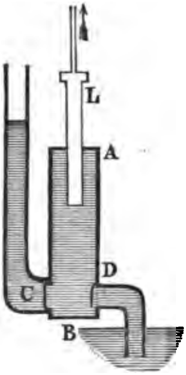
Thus water cannot be raised by a common pump higher than the column of water in a water barometer (say 32 feet). Similarly, mercury cannot be raised more than about 29 inches.

When the length of the tube does not exceed this limit, the exhaustion of air is continued until the liquid passes through B, then through the valve in the piston. Each stroke of the piston now raises the water above it, and causes it to flow out at A.

96. The Force Pump.—In the Force Pump a solid piston or plunger L is moved alternately up and down within a vessel AB, in which there are two valves, one at C opening outwards through which the liquid is forced, the other at D opening inwards for the admission of the liquid.

When the plunger is made to descend, the pressure in AB is increased, the valve D is closed, and some of the liquid is forced through C; when the plunger is drawn up, the pressure in AB is diminished, C is closed, and additional liquid from the reservoir enters at D. It is, of course, necessary that the height of D above the liquid in the reservoir must not exceed the height of the column which can be supported by the pressure of the atmosphere.

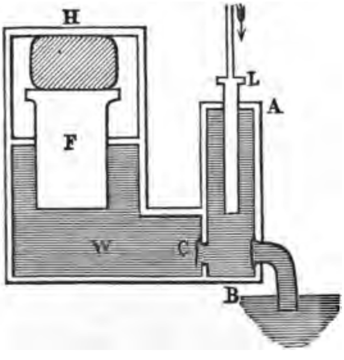




The fire engine is an example of a double force pump, a continuous stream of water being produced by the aid of an air chamber.

97. Bramah's Press.—This machine is used for exerting great pressure.

It consists of a force pump AB, which drives water through C into a vessel W. The pressure thus caused forces up a solid cylinder F, and the substance to be compressed is placed between F and a fixed plate H.



To find the pressure which this machine can exert, let us suppose that the area of the extremity of the plunger L is one square inch, and that of the extremity of F is 400 square inches. While the plunger descends the valve C is open, and therefore the pressure is the same at any two points in the same horizontal plane throughout the liquid.

If, therefore, L is pressed downwards with a force of 100 lbs., the extremity of F will be pressed upwards by a force of 100 lbs. on the square inch (the difference of pressure due to any difference of level being so small that we may disregard it). Thus the whole upward pressure exerted on F is 100 × 400 = 40,000 lbs.

Example for Exercise.—

Find the pressure exerted by a Bramah's Press in which the diameters of the pistons are as 3 to 50, and the applied force is 72 lbs.

Examination on Sections 90—97.

1. Give Boyle's Law. By what other name is it known?
2. Assuming Boyle's Law connecting pressure and volume in a gas, deduce the relation between pressure and density.
3. Describe the construction and use of the barometer.
4. Explain the action of the syphon.
5. Describe the common air-pump, illustrating its action by means of diagrams.
6. What is the principle of Sprengel's air-pump?
7. Describe the condenser.
8. Describe the common pump, showing it cannot raise water above a certain height.
9. Describe the force pump, and Bramah's press.

General Examples in Hydrostatics.

1. If 319 cubic inches of cork weigh as much as 11 cubic inches of iron, compare their densities.
2. The atmospheric pressure being 15 lbs. on the square inch, and the weight of a cubic foot of water being 1000 ounces, find the pressures exerted on a square inch at the following depths below the surface of water :—
    (1) One foot.
    (2) Eighty-five feet.
    (3) Three miles.

3. With the data of last question, if the density of mercury be 13½ times that of water, find the pressure on a square foot at the following depths below the surface of mercury:—

(1) One inch.
(2) Six feet.
(3) One mile.

4. Neglecting the pressure of the atmosphere, compare the pressures at a depth of 17 feet under water, and at a depth of 15 inches under mercury, assuming that their densities are in the proportion of 3 to 34.

5. A vessel, in the form of a cube measuring 3 feet each way, is filled with water; find—

(1) The pressure on its base in vacuo.
(2) The same in air.
(3) Whether the vessel, if just strong enough to contain the water in vacuo, will be able to contain it when air is admitted.

6. A cubic inch of mercury weighs 8 ounces, and the atmospheric pressure is 14½ lbs. on the square inch, find the pressure—

(1) On a square inch 4 inches below the surface of mercury.
(2) On the base of a vessel of mercury one foot square, the mercury being 7 inches deep.

7. A vessel contains mercury, and over the mercury 10 inches depth of water, find, (with the data of examples 2 and 3), to one decimal place, the pressures on a square inch at the following depths below the surface of the water:—

(1) Six inches.
(2) One foot.
(3) Three feet.

8. Find the specific gravities of the following substances, the first weight being that of a portion of the substance, the second that of an equal volume of water:—

(1) Gold; 145 ounces, 7½ ounces.
(2) Diamond; 5¼ grains, 1½ grains.
(3) Ether; 18 drachms, 25 drachms.

9. A lump of lead weighing 1 lb. 6 oz. (specific gravity 11) is suspended in water by a string. Find the tension of the string.

10. Find the specific gravities of the following substances, the

first weight being that of a portion weighed in air, the second that of the same portion weighed in water :—

(1) Lead ; 12 lbs., 10$\frac{16}{17}$ lbs.
(2) Brass ; 16$\frac{2}{3}$ lbs., 14$\frac{2}{3}$ lbs.
(3) Copper ; 31$\frac{1}{2}$ grains, 28 grains.
(4) Ivory ; 6·08 oz., 2·81 oz.

11. Find, to two decimal places, the specific gravities of the following substances which float in water, the first number being the number of cubic inches above, the second that below the surface.

(1) Beech ; 12, 71.
(2) Pine ; 112, 217.
(3) Ice ; 62, 713.

12. A piece of cork floats in water with 40$\frac{1}{2}$ cubic inches above the surface. If the specific gravity of cork be $\frac{1}{4}$, find—

(1) The whole volume of the cork.
(2) How many cubic inches would be above the surface if the cork were caused to float in mercury (*s.g.* 13$\frac{1}{2}$).

13. The specific gravity of sea-water is 1·025. A piece of metal weighing 36·51 lbs., weighs 32·11 lbs. in distilled water. What does it weigh in sea-water?

14. A piece of gold, weighing 5 lbs. in air and 4·74 in water, has the following weights :—

(1) In sulphuric acid, 4·52.
(2) In hydrochloric acid, 4·68.
(3) In alcohol, 4·79.

Find the specific gravities of these liquids to two decimal places.

15. Find the specific gravity of a body of 16 cubic centimetres volume which weighs 100 grammes.

16. A body, weighing 26 grammes, floats in water with two-thirds of its bulk submerged ; find its volume.

17. A certain body A is observed to float in water with half its volume submerged, and when attached to another B of twice its own volume, the combined mass is just submerged. Find the specific gravities of A and B.

18. A piece of platinum weighing 15 lbs. is attached to a piece of iron weighing 10 lbs., and the whole is found to weigh 1 lb.

in mercury (*s.g.* 13½). The platinum by itself weighs 6 lbs. in mercury; find—

(1) The specific gravity of platinum.
(2) " " iron.
(3) The weight in water of the piece of platinum.
(4) " " " iron.

19. A body A, whose specific gravity is ¾, is attached to a body B of equal volume, and the two together just float in water; find the specific gravity of B.

20. A cubic foot of cork is made to float in water with its top horizontal, and the base is then found to be 2·9 inches below the surface. Find its specific gravity.

21. A cubic foot of iron (*s.g.* 7½) floats in mercury (*s.g.* 13½). How many cubic inches are above the surface?

22. A cubic foot of water weighs 1000 oz., and a cubic inch of zinc weighs 4 oz. Find the specific gravity of zinc.

23. A small vessel, exactly filled with distilled water, weighs 530 grains, a small stone weighing 26 grains is thrown in, and an equal bulk of water is thus forced out. The vessel now weighs 546 grains. Find the specific gravity of the stone.

24. A stone of specific gravity 2·5, weighing 5 cwt., is sunk in water. What force is required to raise it?

25. A body weighs 25 lbs. in air, 20 in water, 21 in alcohol. Find the specific gravity—

(1) of the body.
(2) of the alcohol.

26. The weight of a cubic inch of mercury being half a lb., find the pressure on a square inch at the following depths below the surface of the mercury in the tube of a barometer:

(1) 5 inches.
(2) 27 inches.
(3) 31 inches.

27. If the height of the mercurial barometer be 29·6 inches, what is the height of the water barometer?

28. A piece of cork (*s.g.* ¼) weighing 1 lb. is attached to 34 lbs. of lead. The whole weighs 28 lbs. in water. Find the specific gravity of the lead.

29. A block of iron weighs 4·65 lbs. in air, 4 lbs. in water, and 3·85 lbs. in oil. Find the specific gravity of the oil.

30. A piece of wood weighs 6 lbs. A block of lead weighing 17 lbs. is attached to it, and the whole mass weighs 14 lbs. in water. Find the specific gravity of the wood, that of the lead being $11\frac{1}{3}$.

31. A cylindrical cork floats in water with one inch above the surface and $\frac{3}{10}$ of an inch below. What is its specific gravity?

32. A piece of oak (*s.g.* ·74) of 32 cubic inches floats in water. How much water does it displace?

33. If the weight of a cubic inch of mercury be $7\frac{4}{5}$ oz., what is the pressure of the air on a square inch, when the barometer stands at $29\frac{1}{2}$ inches?

34. Two bodies, which weigh 9 lbs. and 10 lbs. in air, weigh each 4 lbs. in water, compare their densities.

35. A piece of silver weighing 1 lb. appears to lose ·095 lb. weight when surrounded by water. Find its specific gravity to two decimal places.

36. An ice-berg floats with 1000 cubic feet above the surface of the sea. Find its volume, assuming its specific gravity to be ·925, and that of the sea 1·025.

37. A mercurial barometer is constructed under 27 feet of water. How high will the mercury stand in it when the reading of a barometer at the surface is 29 inches?

38. A barometer in which a little air has got into the upper part is found to record 28 inches when it should record 30. If the volume of the space above the mercury be $7\frac{1}{2}$ cubic inches, what would be the volume of the air within it at the atmospheric pressure?

39. In a Bramah's press the piston of the force pump is pressed down by means of a lever, the length from the fulcrum to the end of the piston-rod being 3 inches, and from the fulcrum to the hand 21 inches. The diameter of the small piston is 1 inch, that of the large piston 8 inches. How many tons will the machine raise if a force of 20 lbs. be applied to the handle of the lever?

40. A man finds that he can just float in fresh water (a cubic foot of which weighs 1000 oz.), and that his body displaces 3 cubic feet of water;—

(1) How many cubic feet would he displace when floating in sea water of specific gravity 1·025?

(2) What weight could he now bear (clear of the water), so as again to have his whole body just immersed?

41. Assuming the specific gravity of mercury to be 13·566, find the height of a barometer formed of a liquid whose specific gravity is 1·5, the height of the mercurial barometer being 30 inches.

42. If the diameter of the smaller piston of a Bramah's press be diminished one-half, what effect is produced on the magnitude of the force which the engine is capable of exerting?

43. The diameter of the two cylinders of a hydraulic press are 25 inches and 1 inch respectively, the mechanical advantage of the lever employed is 14; calculate the mechanical advantage of the press.

UNIVERSITY EXAMINATION PAPERS.

I.

UNIVERSITY OF CAMBRIDGE.

General Examination for the Ordinary B.A. Degree.

Mechanics and Hydrostatics.

1. Show that forces may be represented in magnitude and direction by straight lines, and give one illustration.

2. Find the magnitude and line of direction of the resultant of two parallel forces acting in the same direction, and deduce from your results the corresponding proposition when the forces act in opposite directions.

3. (1) Determine the force which, acting parallel to an inclined plane, will just support a given weight placed upon it. Find also the pressure on the plane.

(2) Two weights W and 2W are in equilibrium when connected by a string passing over the common vertex of two inclined planes. Given the inclination of the plane on which W rests, show how the inclination of the other plane may be determined by a geometrical construction.

4. What is meant by the moment of a force about a point? Show how a moment may be represented geometrically.

5. Find the ratio of the power to the weight in that system of pulleys in which each string is attached to the weight.

6. (1) Define the term centre of gravity; and find that of a uniform straight rod.

(2) A pencil rests on a table with five-twelfths of its length projecting beyond the edge. A beetle, whose weight is one-fourth of that of the pencil, crawls along it. How far may it crawl without upsetting it?

7. (1) Describe the common balance, stating the requisites of a good balance, and how they are obtained.

(2) How may the true weight of a body be found with a balance in which the arms are unequal?

8. Define a fluid, and distinguish between a liquid and a gas.

9. Show that the surface of a liquid at rest is a horizontal plane.

10. A cylinder open at the top is inverted and immersed vertically in water, find the depth of the vessel under water when half full.

11. (1) Define specific gravity, and show how to determine the specific gravity of a solid body by means of the hydrostatic balance, the specific gravity of the solid being greater than that of the liquid in which it is weighed.

(2) A ball of gutta percha, 2 inches in diameter, encloses a ball of cork 1 inch in diameter, and floats in water. Specific gravity of gutta percha is ·98, of cork ·24. Find what proportion of the volume of the ball will float above the surface of the water.

12. Describe the common barometer. If the tube be not exactly vertical, will the indications of the instrument be incorrect?

II.

UNIVERSITY OF OXFORD.

Second Public Examination—Pass School.

Elements of Mechanics.

1. (1) Explain how uniform velocities are measured.

(2) Two men, A and B, run a race, the former having a start of *a* yards. If the speed of B be to that of A as

3 to 2, find how many yards A will have run when B overtakes him.

2. Enunciate and prove the principle of the parallelogram of forces.

3. (1) How is the parallelogram of forces verified by experiment?

(2) At what angle must two forces, $a^2 - b^2$, $2ab$ act so that their resultant may be $a^2 + b^2$?

4. Find the magnitude of the resultant of two forces of 13 and 35 lbs., when the angle between their directions is 60°.

5. (1) Define the moment of a force about a point.

(2) Prove that the moments of two component forces about any point situated upon the direction of their resultant are equal.

6. Find the velocity which a heavy body will acquire by falling freely through a height of $3\frac{1}{25}$ feet ($g = 32{\cdot}2$).

7. (1) Enunciate Newton's laws of motion.

(2) Two bodies start together from the same point, and move uniformly in directions at right angles to each other; one body moves at the rate of $2m$ feet per second, and at the end of t seconds the distance between them is $(1 + m^2)t$ feet; find the velocity of the other body.

8. (1) Define specific gravity, and show how to find the specific gravity of a mixture of two fluids.

(2) Equal volumes of two fluids, whose specific gravities are 1 and 1·7, contract when mixed by one-tenth of their original volume. Find the specific gravity of the mixture.

9. Explain the action of a syphon, and of a force pump.

10. (1) Explain the principle and construction of the hydrostatic press.

(2) If the diameters of the pistons or plungers be 1 and 9 inches, if the lever be jointed to the rod of the smaller plunger at 6 inches distance from the lever's fulcrum, and if the workman press on the lever with the weight of his body, 140 lbs., applied at 32 inches distance from the fulcrum, find the pressure in tons with which the larger plunger is forced upwards.

III.

UNIVERSITY OF EDINBURGH.

Medical Degree—Preliminary Examination.

Elements of Dynamics (Mechanics).

Kinematics.

1. (1) Define uniform velocity.
   (2) A railway train, moving uniformly, travels 37½ miles in an hour. Find its velocity in feet per second.

2. A body falls from rest, what is its velocity, after falling for two seconds and a half, and what distance will it fall in that time?

Kinetics.

3. (1) Define mass and momentum.
   (2) A cricket ball weighs 6 ounces, and is moving with a velocity of 40 feet per second. Compare its momentum with that of a cannon ball which weighs 20 lbs., but is moving with a velocity of 18 feet per second.

4. Enunciate the second law of motion. Show that it enables us to investigate the motion of a falling body.

5. Find the resultant of the following forces:—5 northwards, 5 eastwards, 1 westwards, and 2 southwards.

Statics.

6. Parallel forces of 4 and 6 pounds act in the same directions at two points, A and B, in a body. Find at what point in AB a single force must act to maintain equilibrium.

7. State the three kinds of equilibrium, and give an example of each.

8. A horizontal force of 5 lbs. supports a weight of 12 lbs. on an inclined plane. Find the pressure on the plane.

Hydrostatics.

9. (1) Define a fluid.
   (2) What is the law of pressure at different depths below the surface of a liquid?

10. The specific gravity of lead is 11. What is the real weight of a mass of lead which, when suspended in water, seems to weigh 1 lb?

IV.

UNIVERSITY OF LONDON.

Matriculation Examination.

Natural Philosophy.

1. (1) State the parallelogram of forces. Explain the meaning of the terms employed in your statement.
   (2) Apply it to show that if four forces acting on a point be represented by the sides of a rectangle taken in order, they will be in equilibrium.
2. (1) Explain the meaning of the words composition and resolution of forces, and show how forces may be compounded and resolved.
   (2) A particle is acted upon by a force whose magnitude is unknown, and whose direction makes an angle of 60° with the horizon. The horizontal component of the force is known to be 1·35. Determine the total force and also its vertical component.

3. A substance is weighed from both arms of an unequal balance, and its apparent weights are 9 lbs. and 4 lbs. Find the ratio between the arms.

4. A balloon is carried along by a current of air moving from east to west at the rate of 60 miles an hour, having no motion of its own through the air, and a feather is dropped from the balloon. What sort of a path will it appear to describe as seen by a man in the balloon?

5. Suppose that at the equator a straight hollow tube were thrust vertically down towards the centre of the earth, and that a heavy body were dropped through the centre of such a tube. It would soon strike one side. Find which, giving a reason for your reply.

6. A body weighs in air 80 grains, in water 56 grains, and in another liquid 46 grains. Find the specific gravity of this liquid.

7. What becomes of the weight which the body, in the last question, appears to lose?

8. A cylindrical bell 4 feet deep, whose content is 20 cubic feet, is lowered into water until its top is 14 feet below the surface of the water, and air is forced into it until it is three-quarters full. What volume would the air occupy under the atmospheric pressure—the water barometer being at 34 feet?

State the principles in which your answer is based.

[*This Examination also contains Questions in Optics.*]

V.

UNIVERSITY OF GLASGOW.

Preliminary Medical Examination.

Elements of Mechanics.

1. (1) Explain how a force may be represented by a straight line.
   (2) Two forces of 30 and 40 lbs. respectively act upon a particle in directions at right angles to each other, find the magnitude of their resultant in pounds.
2. (1) What is a "lever"?
   (2) Distinguish the three kinds of simple levers, and point out (with reasons), to which class belongs a pair of nutcrackers, a pair of tongs, a pair of scissors.
3. (1) Explain the terms "velocity," "acceleration;" and when the acceleration is constant what is the connection between it and velocity?
   (2) A body under a constant acceleration moves from rest over 10 feet in the first second, how far has it moved at the end of the second second?

4. Find the ratio of the power to the weight in a system of pulleys, in which each pulley hangs by a separate string.

5. When a cannon is fired, explain carefully what takes place. If the cannon be free to move, find its velocity on being fired, supposing it to weigh 200 lbs., the bullet weighing 5 lbs., and being projected horizontally with a velocity of a mile in five seconds.

VI.

UNIVERSITY OF CAMBRIDGE.

Local Examination.

Mechanics.

1. (1) Show how a straight line may be made to represent the several elements which completely determine a force.
   (2) Three forces, of 3, 7, and 10 lbs. respectively, are in equilibrium at a point. How are they directed?
2. (1) Enunciate the proposition known as the parallelogram of forces.
   (2) The resultant of two forces acting at an angle of 120° is perpendicular to the smaller component. The greater component is a force of 100 lbs. Find the other component and the resultant.
3. A heavy rod hangs vertically and is free to turn about its upper end. A horizontal force equal to half the weight of the rod is applied at the lower end. Find the inclination to the horizon when there is equilibrium.
4. (1) Find the resultant in magnitude and direction of two parallel forces acting in the same direction.
   (2) Weights of 6, 8, and 10 lbs. respectively, are placed on the middle points of the edges of a triangular table. Find how much the pressures on each of the three legs are increased thereby.
5. (1) Define the "centre of gravity" of a body, and find that of a plane triangular area.

(2) Weights of 1, 2, 3, 4 lbs. respectively, are attached to the corners of a light square frame. Determine the point by which the frame may be suspended in any position.

6. (1) Find the relation of the power to the weight in the wheel and axle.

(2) Find the radius of the axle when a weight of 16 lbs. just balances a weight of 1 cwt., the radius of the wheel being 21 inches.

7. A horse, in drawing a load up a steep hill, takes a zigzag course, and a man going down the same on horseback leans backward; explain the advantage in each case.

ANSWERS.

CHAPTER I.—PRELIMINARY.

Examination on Chapter I.

[The Answers to the Examination Questions will all be found in the text of the book. The form of the question has been occasionally changed in order to prepare the Student for the various ways in which the same question may be set by different examiners.]

CHAPTER II.—KINEMATICS.

Examples for Exercise.

§ 4.

1. $\dfrac{2 \times 1760 \times 3}{9 \times 60} = 19\tfrac{5}{9}$.

2. The former velocity is $26\tfrac{2}{23}$, the latter is 27 and therefore the greater.

3. 1118 feet = distance travelled in 1 sec.

$\therefore$ 1 „ = „ „ „ $\dfrac{1}{1118}$ sec.

$\therefore$ $43 \times 1760 \times 3$ ft. = „ „ $\dfrac{43 \times 1760 \times 3}{1118} = 203\tfrac{1}{13}$ sec.

§ 5.

1. $s = 100 \times 60 \times 60$ ft. = 360,000 ft. = $68\tfrac{2}{11}$ miles.

2. $v = \dfrac{1511460 \times 1760 \times 3}{27 \times 24 \times 60 \times 60} = 3421$.

3. $t = \dfrac{b}{a}$.

§ 7.

1. 48·3.

2. Find the average velocity for the last two seconds. 64·4.

3. $\dfrac{a+b+c}{3}$.

4. $v = \dfrac{s}{t} = \dfrac{s_1 + s_2}{t_1 + t_2} = \dfrac{v_1 t_1 + v_2 t_2}{t_1 + t_2}$.

§ 11.

1. It is convenient to take a mile an hour as the unit of velocity in this example.

Velocity up stream $= 8 - 2 = 6$ miles an hour.
" down " $= 8 + 2 = 10$ " "

2. Take one foot a minute as the unit of velocity.

$$v_1 = \text{velocity of sledge} = \frac{12\times1760\times3}{24\times60} = 44.$$

$$v_2 = \text{velocity of floe} = 45.$$

$$\therefore v = 44 - 45 = -1.$$

Or the sledge moves southward at the average rate of one foot per minute.

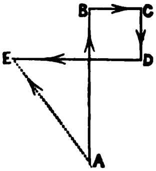
3. Let v_1 = velocity of steamer in miles per hour.
v_2 = " river " "

Then we obtain the simultaneous equations—

$$v_1 + v_2 = 15,$$
$$v_1 - v_2 = 11.$$

Solving these we find $v_1 = 13$, $v_2 = 2$.

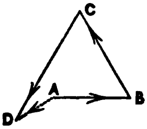
§ 14.



1. Drawing the velocities AB, BC, CD, DE, we obtain the resultant AE = 5.

The student will observe that two of the velocities intersect, hence the figure is not strictly speaking a polygon. Nevertheless the proof in § 14 includes such cases.

2. Drawing the velocities AB, BC, CD, we obtain the resultant AD. It is a little greater than $1\frac{2}{3}$. The advanced student will be able to prove geometrically that AD = $\sqrt{3}$, and the angle DAB = 150°.



§ 15.

1. $v^2 = v_1{}^2 + v_2{}^2 = 169 \quad \therefore v = 13.$

2. $v^2 = (10{\cdot}5)^2 + (3{\cdot}6)^2 = 123{\cdot}21. \quad \therefore v = 11{\cdot}1.$

3. Since the stream is 972 feet broad, and the man swims across 3·6 feet in each second, he will arrive at the other side in t seconds if—

$$t = \frac{972}{3{\cdot}6} = 270.$$

and during that time he will have been carried down $270 \times 10{\cdot}5 = 2835$ feet.

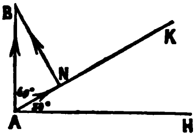
§ 17.

1. In figure 1, let AB = 7, BC = 8. Then BE = 4.
$\therefore \mathrm{AC}^2 = 49+64+2\times4\times7 = 169. \quad \therefore \mathrm{AC} = 13.$

2. In figure 2, let AB = 5, BC = 8. Then BE = 4.
$\mathrm{AC}^2 = 25+64 - 2\times4\times5=49. \quad \therefore \mathrm{AC} = 7.$

3. (1) $\sqrt{2+\sqrt{3}}$, (2) $\sqrt{2+\sqrt{2}}$, (3) $\sqrt{2-\sqrt{3}}$.

§ 18.



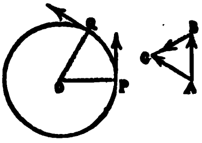
1. AN is the component required, and is equal to 2, since BAN is the half of an equilateral triangle.

2. Taking the figure in § 18, let the angle CAB = 45°. Then $\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2 = 2\,\mathrm{AB}^2$.

$$\therefore \mathrm{AB} = \frac{1}{\sqrt{2}}\mathrm{AC}, \text{ and if } \mathrm{AC} = 3\sqrt{2}, \mathrm{AB} = 3.$$

§ 20.

1. From any point A draw AB, AC to represent the velocities at P and Q, complete the $\triangle$ABC. The third side BC is the change in velocity, and is equal to *one*. Its direction makes an angle of 120° with that of the first velocity.



2. In the same way we find the change of velocity $= \sqrt{2}$ in a direction at an angle of 135° to the first velocity.

ANSWERS.

§ 22.

1. $v = 13, s = 120.$
2. $t = 5, v = 50.$
3. $s = 2093, v = 370{\cdot}3.$

§ 23.

1. (1) $34\frac{1}{2}$ feet.
   (2) 7.
   (3) In $5\frac{1}{3}$ seconds.
   (4) $42\frac{2}{3}$ feet.
   (5) $v = -8, s = 32.$
   (6) In $10\frac{2}{3}$ seconds, $v = -16.$
2. (1) $v = V - gt = ng - ng = 0.$
   (2) $s = Vt - \frac{1}{2}gt^2 = n^2g - \frac{1}{2}n^2g = \frac{1}{2}n^2g.$
   (3) Putting $t = 2n, s = 2n^2g - 2n^2g = 0.$
   (4) „ $t = 2n, v = ng - 2ng = -ng.$

General Examples in Kinematics.

1. (1) $17\frac{3}{5}$. (2) $152\frac{7}{8}$. (3) $\frac{1}{1120}$. (4) $\frac{5}{12}$. (5) $97\frac{7}{9}$. (6) 15.
   (7) 50. (8) $\frac{1}{96}$. (9) 976800000. (10) 1.
2. (1) 15. (2) $81\frac{9}{11}$. (3) 765. (4) 915. (5) 68925.
3. (1) 5. (2) $27\frac{3}{11}$. (3) 253. (4) 305. (5) 22975.
4. (1) 2·2. (2) 12. (3) 112·2. (4) 134·2. (5) 10109.
5. (1) 1. (2) $\frac{11}{60}$. (3) $\frac{1}{51}$. (4) $\frac{1}{61}$. (5) $\frac{1}{4595}$.
6. (1) $\frac{a}{b}$. (2) $\frac{2}{3}$. (3) $\frac{a}{20b}$. (4) $\frac{10}{a+b}$. (5) $\frac{a^2b}{ab^2} = \frac{a}{b}$.
   (6) $\frac{a^2 - b^2}{a+b} = a - b.$
7. (1) 2π. (2) $\frac{2\pi a}{v}$. (3) 1. (4) $4\frac{2}{3}$.
8. (1) $4\frac{8}{9}$. (2) $3\frac{1}{3}$.
9. (1) $5\frac{5}{21}$. (2) $3\frac{4}{7}$.
10. One tenth of a second.
11. 35 miles.

12. (1) 27 forwards. (2) 23 backwards. (3) 2 forwards. (4) 6 backwards. (5) $2b$ forwards. (6) $6x^2$ backwards.

13. $1\frac{53}{57}$ hours = 1 hour 55 min. $47\frac{7}{19}$ sec.

14. $9\frac{9}{22}$ miles = 9 miles 2160 feet.

15. 125 : 88.

16. (1) 10. (2) $1\frac{1}{4}$. (3) 17. (4) $5\frac{1}{12}x$. (5) 29. (6) Solving the simultaneous equations, we get $v_1 = 12$, $v_2 = 5$. Hence $v = 13$. (7) $v_1 = \frac{1}{2}(a^2 - b^2)$, $v_2 = ab$; $v = \frac{1}{2}(a^2+b^2)$.

17. The answers found by measurement are approximately :—

(1) $6\frac{3}{4}$. (2) $6\frac{1}{3}$. (3) $4\frac{1}{3}$. (4) $4\frac{1}{2}$. (5) $7\frac{1}{3}$.

18. (1) 19. (2) $1\frac{6}{7}$. (3) $3x^2+3x+1$.

19. (1) 19. (2) $9\frac{1}{4}$. (3) x^2+x+1.

20. (1) 1. (2) 7.

21. (1) 1. (2) $\sqrt{8} = 2\sqrt{2}$. (3) $\sqrt{149}$.

22. 1300.

23. $17\frac{7}{8}$ feet per second.

24. 29 miles.

25. This is the converse of § 16.

26. 120°.

27. 13.

28. 5.

29. (1) 1 north. (2) 9 south. (3) 1 north. (4) $\sqrt{2}$ south-west.

30. The third side of the triangle.

31. (1) 10. (2) 20. (3) 48.

32. $\sqrt{27} = 3\sqrt{3}$.

33. (1) 10. (2) 30 feet. (3) 2.

34. (1) 19. (2) $11\frac{1}{2}$. (3) 115 feet.

35. (1) 16 feet. (2) 48 feet. (3) 176. (4) 484 feet.

36. (1) $6\frac{1}{2}g$. (2) $18g$ feet.

37. (1) $5g = 161$. (2) $12\frac{1}{2}g = 402{\cdot}5$ feet. (3) $ng = 32{\cdot}2n$. (4) $\frac{1}{2}n^2g = 16{\cdot}1n^2$. (5) 4 seconds. (6) 6 seconds.

38. (1) 8. (2) $18\frac{1}{2}$. (3) $129\frac{1}{2}$. (4.) In $9\frac{2}{3}$ seconds. (5) $140\frac{1}{6}$ feet.

39. (1) After $n - r$ sec. $v = V - gt = ng - (n-r)g = rg$.

„ $n + r$ „ $v = V - gt = ng - (n+r)g = -rg$.

(2) $s = Vt - \frac{1}{2}gt^2$, which reduces to $\frac{1}{2}(n^2 - r^2)g$ whether we substitute for t the value $n-r$ or $n+r$.

CHAPTER III.—KINETICS.

Examples for Exercise.

§ 25.

1. $\frac{4 \times 1000}{100} = 40.$

2. The whole momentum is 72.

∴ The momentum of each part is 24.

∴ The velocities are 24, $\frac{24}{2}$, and $\frac{24}{3}$, or 24, 12, and 8.

§ 26.

1. Momentum generated in two seconds $= 5 \times 13 = 65$.

∴ Momentum generated in one second $= 32\frac{1}{2}$.

∴ Force required $= 32\frac{1}{2}$.

2. Momentum is here increased from 32 to 80.

∴ Momentum generated $= 48$.

∴ A force 6 will generate this momentum in 8 seconds.

§ 27.

Here a weight of 196 lbs. has to be raised 3000 feet.
∴ Work $= 196 \times 3000 = 588{,}000$ foot-pounds.

§ 36.

Let $v =$ velocity generated in one second in either mass.
Then $f_2 = 12v = 8$.

$$\therefore v = \frac{8}{12} = \frac{2}{3}, \text{ and } f_1 = 9v = 6.$$

§ 38.

1. $mv = mgt = 2318{\cdot}4$.
$s = \frac{1}{2}gt^2 = 5216{\cdot}4$.

2. $s = Vt + \frac{1}{2}gt^2$.

∴ $1000 = 2V + \frac{1}{2} \times 32{\cdot}2 \times 4 = 2V + 64{\cdot}4$.

∴ $V = 467{\cdot}8$.

The force is measured by the initial momentum—

$$mV = \tfrac{1}{8} \times 467{\cdot}8 = 58{\cdot}5.^*$$

---

* Since the value of g varies at different heights in any one place, the value 32·2 is only approximate; and, in all calculations founded on that

3. To find the time of falling—

$$s = \tfrac{1}{2}gt^2. \quad \therefore t^2 = \frac{2s}{g} = \frac{1118}{32{\cdot}2} = 34{\cdot}7.$$

$$\therefore t = \sqrt{34{\cdot}7} = 5{\cdot}9.$$

To find the time the sound of the splash takes to reach the ear, we employ formula (3) of uniform velocity (*See* § 5)—

$$t = \frac{s}{v} = \frac{559}{1118} = {\cdot}5.$$

$$\text{Time required} = 5{\cdot}9 + {\cdot}5 = 6{\cdot}4.$$

§ 39.

From (19) $s = Vt - \tfrac{1}{2}gt^2$.

when $s = 0$, $Vt - \tfrac{1}{2}gt^2 = 0$.

$$\therefore V - \tfrac{1}{2}gt = 0.$$

$$\therefore V = \tfrac{1}{2}gt = \tfrac{1}{2} \times 32{\cdot}2 \times 4 = 64{\cdot}4.$$

Substituting this value for V, when $t = 1$, $s = 48{\cdot}3$; $t = 2$, $s = 64{\cdot}4$; $t = 3$, $s = 48{\cdot}3$.

§ 41.

1. By § 13 the resultant velocity is zero. Therefore the forces are in equilibrium. (*Erratum.*—In the question, for *produce* read *be in.*)

2. By § 17—

$$f^2 = (3\tfrac{1}{2})^2 + 4^2 + 2 \times 3\tfrac{1}{2} \times 2,$$

$$= 12\tfrac{1}{4} + 16 + 14 = 42\tfrac{1}{4} = \frac{169}{4}.$$

$$\therefore f = \frac{13}{2} = 6\tfrac{1}{2}.$$

3. Here the angle between the two directions is 60°. The figure is the same as that in the answer to Example 1, § 18.

AB = 12 ∴ AN the resolved force is 6.

---

value, it is best to omit all figures beyond the first decimal place. In the present case the result of dividing gives 58·475, and as this number is nearer in value to 58·5 than to 58·4, it is better to write the former as the approximate answer. The answers to subsequent questions are formed on this principle.

§ 43.

Here $M = 10$, $V = 14$, $m = \frac{1}{8}$. $\therefore v = 1120$.

General Examples in Kinetics.

1. (1) 178. (2) 0.
2. 1500.
3. 5 seconds.
4. 672,000 foot-pounds.
5. No work is done against gravity.
6. The momentum generated by the weight of one ounce in one second is $\frac{1}{16} g$. $\therefore mv = \frac{1}{16} g$.

   But $m = 2$. $\therefore v = \frac{1}{32} g = 1$.
7. $s = 402{\cdot}5$, $v = 161$.
8. 742·4 feet.
9. 2·5 seconds.
10. 197·2 feet.
11. 680·2 feet.
12. 35 seconds, 19722·5 feet.
13. The expression "a force of two pounds" is not strictly correct, but must be considered to mean "a force equal to the weight of two pounds," that is, a force of $2g$ or 64·4. (See § 26.)

    Thus the momentum generated in a second is $2g$, and therefore the mass of 10 lbs. receives a velocity of $\frac{2g}{10} = 6{\cdot}4$.
14. 16·1.
15. 402·5 feet.
16. 161 ; $402\frac{1}{2}$ feet. (Compare Example 7.)
17. After 2 seconds at a height of 95·6 feet.
    After 4 „ „ 62·4 „
    After 6 „ 99·6 feet below the point of projection.
18. 24·6.
19. (1) 6·2 seconds.

    (2) Since the momentum is the same as in the last case, the initial velocity will be 200 feet. Answer, 12·4 seconds.
20. 983·9 feet.
21. Complete the parallelogram ABDC, and produce the

diagonal DA its own length (backwards) to E. AE is the line required, and makes an angle of 155° with AB.

22. The resultant force is 5.

23. 5.

24. $10\sqrt{2}$ or 14·14 approximately. Direction NE.

25. 13 lbs.

26. 7·8125 or $7\frac{13}{16}$.

27. 5.

28. Since action and reaction are equal and opposite the momentum given to one part forwards is equal to that given to the other part backwards. Thus the whole momentum is unchanged, and is equal to that of the two parts.

$$\therefore mv = 12 \times 700 + 20 \times 380 = 16{,}000.$$

29. The whole momentum is unchanged, and since one part is reduced to rest, the other part moves on with the original momentum of the shell $= 20 \times 600 = 12{,}000$.

30. The two equal parts receive equal and opposite changes of momentum, and therefore of velocity. The change of velocity given to the first part is 40, therefore that given to the second is -40. Its new velocity is therefore $80 - 40 = 40$.

31. 72 lbs.

32. 43 lbs.

33. The formula $s = \frac{1}{2} gt^2$ gives us $t = \frac{10}{23}$, $\therefore v = gt = 14$.*

34. (1) 73.
    (2) $\sqrt{(a^2 + b^2)}$.
    (3) $a^2 + b^2$.

35. Substituting in equations (11) and (12), § 23, we get

$$0 = 10 - at,$$
$$10 = 10t - \tfrac{1}{2} at^2.$$

Putting $t = \frac{10}{a}$ in the second equation, we find $a = 5$.

The force required is therefore a force which in one second generates a velocity of 5 in the body; or if *m* be the mass of the body, the force is $5m$.

---

* If in the formula $s = \frac{1}{2}gt^2$, we substitute for t the value $\frac{v}{g}$ obtained from the formula $v \quad gt$, we obtain the equation $v^2 = 2gs$, by which such questions as this may be directly solved.

36. It follows from the parallelogram of forces that the resultant of two forces is greatest when they act in the same direction along the same straight line, and least when in opposite directions.

$$\therefore F_1 + F_2 = 12.$$
$$F_1 - F_2 = 6.$$

from which we get $F_1 = 9$, $F_2 = 3$.

37. Forming the triangle of forces, we find that it is half an equilateral triangle, and the angle at which P and Q act is its exterior angle = 150°.

38. This follows from the fact that the angle in a semicircle is a right angle.

CHAPTER IV.—STATICS.

Examples for Exercise.

§ 45.

1. The triangle of forces is equilateral. Its exterior angles are each 120°.

2. The tensions of the strings and the weight form 3 forces in equilibrium. Their directions are at 120° to each other. Therefore the 3 forces are equal. (Compare last example.)

3. Any two sides of a triangle are greater than the third.

§ 49.

1. 12 lbs., 10 inches from greater weight.

2. 12 lbs., 5 inches from the greater force.

3. If they are not parallel two of them meet in a point, and the third must, to be in equilibrium, pass through that point.

§ 50.

1. A force 3, 14 inches beyond the greater force.

2. A force 3, 8 inches beyond the greater force.

§ 53.

Let ABCD be the square, the weights of 1 and 3 lbs. are equivalent to a weight of 4 lbs. acting at E in AB, EB = $\frac{1}{4}$ AB, the weights of 9 and 3 lbs. are equivalent to a weight of 12 lbs. at a point F in CD, CF = $\frac{1}{4}$ CD. Join EF, the centre of gravity of 4 lbs. at E and 12 lbs. at F is G in EF, FG = $\frac{1}{4}$ EF.

§ 55.

The perpendicular drawn from the centre of gravity must fall within the base.

§ 57.

1. Moment of force at A = 0, since perpendicular = 0.
„ „ B = 2 × $\frac{1}{3}$ of 10 = 6$\frac{2}{3}$.
„ „ C = 3 × $\frac{2}{3}$ of 10 = 20.
„ „ D = 4 × 10 = 40.

2. Let AB, AD be the forces, and let E be any point in their resultant. Then the moment of AB about E is twice the triangle ABE, and is therefore equal to the parallelogram on AB, as base, formed by drawing a parallel to AB through E. Similarly the moment of AD about E is equal to the parallelogram on AD as base, formed by drawing a parallel to AD through E. These two parallelograms are easily proved to be equal by the help of Euclid I., 43.

General Examples in Statics.

1. (1) R = 17; (2) Q = 8; (3) R = 10; (4) Q = 7.

2. In the triangle of forces the angle opposite P is the greatest, and that opposite R is the least. But the angles mentioned in the question are the exterior angles of the triangle.

3. An equal force. (See Examples § 45.)

4. Six inches from the greater weight.

5. (1) R = 9 lbs., AC = 8 in.; (2) R = 11 lbs., AC = 1 ft.; (3) R = 20 lbs., AC = 2 in.

6. (1) R = 5 lbs., AC = 2 ft.; (2) R = 2 lbs., AC = 3 ft.;
   (3) R = 10 lbs., AC = − 1 ft.

7. A bears 120 lbs., B bears 80 lbs.

8. Half an inch.

9. One third along the line drawn from the centre of the square to bisect the middle rod.

10. Ten inches from the extremity on which the weight is placed.

11. Three inches from A.

12. Nine inches from A.

13. The middle point of the axis of the hat.

14. Since the diagonals bisect each other, each diagonal contains the centres of gravity of the two triangles into which the other divides the parallelogram. Therefore the centre of gravity of the whole parallelogram lies in each diagonal. It can therefore only be their point of intersection.

15. In the wire parallelogram the centre of gravity of either pair of parallel sides is the point O found in § 52 (2).

16. Each has been shown to pass through the centre of gravity.

17. (1) In a quadrilateral, the centre of gravity of the whole is in the line joining the centres of gravity of the two triangles into which it is divided. But this may be done in two ways. Thus we obtain two lines, each of which contains the centre of gravity. Therefore the centre of gravity is at their intersection.

(2) Any rectilineal figure may be divided into triangles. Find the centre of gravity of each triangle, and conceive a heavy particle equal to its mass placed at that point. Then find the centre of gravity of the system of heavy particles. § 53 (5).

18. The centre of gravity of the loaded table must be vertically over a point within the area formed by joining the extremities of the legs of the table.

19. The first moment is represented by twice the area of the △ ABC, the second by twice the △ ABD.

20. Each of these moments is represented by the area of the square.

21. (1) For simplicity let P act at right angles to AB.
By (23) P × AC = Q × BC. Add to each side P × BC, and we get P × AB = R × BC.
(2) Let D be a point in AB produced.
Then (P × Q) × BD = R × BD. Add these quantities to each side of the equation obtained above and we have—

$$P \times AD + Q \times BD = R \times CD.$$

22. Take the figure of § 41, and let O be a point in the plane, and first let O be above AB and to the left of AD.

Then OAC + ABC = OAB + OBC.
and OBC = OAD + ABC (see § 53, footnote).
∴ OAC = OAB + OAD.
∴ Moment of AC = moment of AB + moment of AD.

In a similar way the proposition is proved for any other position of O. It is important to observe that when the forces tend to turn the body in different directions about O, the algebraical sum of the moments is the difference of the figures which represent the moments.

23. The diameter drawn through *any* angular point, bisects the figure, and therefore by symmetry contains the centre of gravity. Compare § 53 (4).

24. The diameter drawn through any bead must pass through the centre of gravity, which must therefore be the centre of the circle.

25. (1) Stable; (2) stable in one direction, unstable in another; (3) unstable; (4) neutral; (5) stable; (6) stable; (7) neutral in one direction, unstable in another.

CHAPTER V.—MACHINES.

Examples for Exercise.

§ 60.

1. If the 3 lb. weight is at A, AC = $\frac{3}{4}$ AB.
2. $3\frac{1}{3}$ lbs.
3. AC = 4 in.

§ 62.

$W^2 = PQ = 8 \times 12\frac{1}{2} = 100. \quad \therefore W = 10$ lbs.

§ 64.

Here P is 600 lbs, $AC = 6$, $BC = 1$. $\therefore W = 3600$ lbs.

§ 66.

$W = 2^n P = 2^5 \times 3 = 32 \times 3 = 96$ lbs.

§ 67.

The diagram resembles the first figure of § 67, but has 3 pulleys in each block.

§ 68.

$W = 62 = (2^5 - 1) P = 31P. \quad \therefore P = 2$ lbs.

§ 70.

1. To find the height of the plane we have

$$BC = \sqrt{AB^2 - AC^2} = 5 \text{ ft.}$$

If, therefore, 5 ft. represents 5 lbs.
13 ft. „ 13 lbs.

2. Length of plane $= \sqrt{12^2 + 35^2} = 37$.

Thus 37 represents 74 cwt.
12 „ 24 cwt. = 1 ton 4 cwt. = P.
35 „ 70 cwt. = 3 tons 10 cwt. = R.

§ 71.

Here the height is $\sqrt{25^2 - 24^2} = 7$.
Thus 7 represents 14 lbs.
∴ 24 „ 48 lbs.

§ 73.

1. Since $F = \mu R$, we have $10 = \mu 80$. $\therefore \mu = \frac{10}{80} = \frac{1}{8}$

2. $\mu = \frac{3}{4} = \frac{BC}{AC}$.

If, then, $BC = 3$, $AC = 4$, and $AB = \sqrt{3^2+4^2} = 5$.
But AB is 10 inches.
$\therefore$ BC is 6 inches.

§ 74.

(1) In the lever the points A and B describe arcs of circles, whose radii are AC and BC. Since the arcs are proportional to their radii, if we take kAC to represent the arc described by A, kBC will be the arc described by B, and since by (23)

$$P \times AC = W \times BC,$$
$$\therefore P \times kAC = W \times kBC.$$

Or the work done by P is equal to that done against W.

(2) If, in the figure of § 66, P be raised 1 foot, the strings DA and AP will each be shortened $\frac{1}{2}$ a foot, and A will be raised $\frac{1}{2}$ a foot. Similarly if A be raised $\frac{1}{2}$ a foot, B will be raised $\frac{1}{4}$ of a foot, and C, $\frac{1}{8}$ of a foot.

Thus the path of P is 8 times that of W, or if k represent P's path, $2^3\,k$ represents W's.

$$\text{Now} \qquad W = 2^3 P,$$
$$\therefore kW = 2^3 kP.$$

Or the work done by P is equal to that done against W.

The theorem is proved similarly for any number of pulleys.

(3) This follows directly from (1).

General Examples in Machines.

THE LEVER.

1. (1) AC = 1 ft. $1\frac{1}{2}$ in. (2) AC = $4\frac{4}{5}$ in. (3) AC = 2 in.
2. 2 lbs.
3. 6 oz.

4. 1 ft. 2½ in. from the fulcrum.

5. 3 lbs.

6. (1) A 120 lbs., B 80 lbs. (2) A 275 lbs., B 75 lbs. (3) A 180 lbs., B 120 lbs.

7. 3¾ lbs.

8. 6 feet.

9. If the force of 5 lbs. act at A, and that of 6 lbs. at B, then the fulcrum is at C.

(1) AC = 3 in. C is between A and B.

(2) AC = 33 in. B is between A and C.

10. 40 lbs. and 35 lbs. Regard each peg in turn as the fulcrum.

11. 15 lbs.

12. 7 lbs.

13. ½ a lb.

14. Let x be the weight of the rod, y the distance of its centre of gravity from the first position of the fulcrum. Then we have—

$$3 \times 4 = x \times y,$$
$$5 \times 3 = x\,(y + 1).$$

Subtracting the first equation from the second, we get $x = 3$ lbs., and therefore $y = 4$ ft. Thus the rod weighs 3 lbs., and is 16 feet long.

The Balance.

15. 5 lbs.

16. 5 lbs. 1 oz. and 6 lbs. 4 oz.

17. 13 to 12.

18. (1) 5 lbs. 1 oz. (2) 9 to 8.

The Wheel and Axle.

19. 15 lbs.

20. 50 lbs.

21. 6480 lbs.

Pulleys.

22. The first system with 2 pulleys, and the second system with 4 pulleys.

23. The second system with 7 pulleys, and the third system with 3 pulleys. Observe that in the second system of pulleys

the number of sheaves in the fixed block is either equal to that in the moveable block or exceeds it by one.

24. 2 lbs.

25. 4 pulleys in each block.

26. See figure of § 66. Tension of AD is 10 lbs. $\therefore$ Weight of A + tension of ABE = 20 lbs. $\therefore$ Tension of ABE = 19 lbs. $\therefore$ Weight of B + tension of BCE = 38 lbs. $\therefore$ Tension of BCE = 37 lbs. $\therefore$ Weight of C + W = 74 lbs. $\therefore$ W = 73 lbs.

27. Weight of block + W = 60 lbs. $\therefore$ W = 57 lbs.

28. See figure of § 68. Tension of AD = 10 lbs. $\therefore$ Tension of ABE = 20 lbs. + weight of A = 21 lbs. $\therefore$ Tension of BCF = 42 lbs. + weight of B = 43 lbs. $\therefore$ Tension of CK is 86 lbs. + weight of C = 87 lbs. This tension is equal to W + P + the weight of the 3 pulleys. $\therefore$ W = 87 − 10 − 3 = 74 lbs. (or W may be obtained by adding the three tensions 10 + 21 + 43 = 74 lbs.)

THE INCLINED PLANE.

29. Here the length is to the height as 5 to 3.
$\therefore$ The weight is to the power as 5 to 3.
But the weight is 10 lbs.
$\therefore$ (1) The force is 6 lbs.
(2) The pressure is 8 lbs.

30. Here the base is 63. If 16 represent 80 lbs., 63 represents 315 lbs.

31. $R^2 = P^2 + W^2 = 35^2 + 120^2 = 125^2$. $\therefore$ R = 125 lbs.

32. 10 lbs.

33. 30° to the horizon.

34. (1) 25 lbs. (2) 12 lbs. (3) 16 lbs.

MISCELLANEOUS.

35. The power is $\frac{1}{8}$ of the weight.

36. Let h be the common height, W and W′ the weights, P the tension of the cord. Then P is a force parallel to the plane in each case.

If therefore h represents P,
5 „ W,
And in the same way 6 „ W′.
Or W is to W′ as 5 to 6.

37. The power moves through an arc of a circle four times that traversed by the weight in the same time.

$\therefore$ Its velocity is 28.

CHAPTER VI.—HYDROSTATICS.

Examples for Exercise.

§ 77.

1. $\dfrac{219}{3} = 73.$

2. $\dfrac{812}{1728} = \dfrac{203}{432}$ of a lb.

§ 81.

1. $p = 10 \times 144 = 1440$ lbs.
$p + xq = 20 \times 144 = 2880$ lbs.
$\therefore xq = 1440$ lbs.
But $x = 4 \quad \therefore q = 360$ lbs.

2. $xq = 15 \times 144.$ But $q = 62\tfrac{1}{2}.$
$\therefore x = \dfrac{15 \times 144}{62\frac{1}{2}} = 34{\cdot}56$ ft.

This is the height of the water barometer for a pressure of 15 lbs. on the square inch.

3. Here the additional pressure due to 5 ft. is $2\tfrac{1}{2}$ lbs.
$\therefore$ „ „ 1200 ft. is 600 lbs.
$\therefore$ The whole pressure at 400 yds. is $2 + 600 = 602$ lbs.

§ 86.

1. $\dfrac{540}{62\frac{1}{2}} = 8{\cdot}64.$

2. (1) A ton of water occupies $\dfrac{20 \times 112}{62\frac{1}{2}} = 35{\cdot}84$ cubic feet.

$\therefore$ A ton of ice occupies $\dfrac{10 \times 35{\cdot}84}{9} = 39{\cdot}82$ cubic feet.

(2) A cubic yard of water weighs $27 \times 62\frac{1}{2} = 1687\frac{1}{2}$ lbs.

$\therefore$ A cubic yard of ice weighs $\frac{9 \times 1687\frac{1}{2}}{10} = 1518\frac{3}{4}$ lbs.

3. 16 cubic centimetres of water weighs 16 grammes.

$$\therefore s = \frac{108{\cdot}8}{16} = 6{\cdot}8.$$

§ 87.

1. $s = \frac{14}{4} = 3\frac{1}{2}$.

2. For A, $s = \frac{10}{5} = 2$.

A and B displace 13 lbs of water.

A alone displaces 5 lbs of water.

$\therefore$ B displaces 8 lbs. of water.

$$\therefore \text{for B, } s = \frac{6}{8} = \frac{3}{4}.$$

3. The water displaced by the cork and silver weighs 6 lbs.
" " cork alone " 4 lbs.
$\therefore$ " " silver " 2 lbs.

$$\therefore s = \frac{21}{2} = 10\frac{1}{2}.$$

§ 88.

1. For the body, $s = 5$.

For the liquid, $s = \frac{5 - 3}{5 - 4} = 2$.

2. The weight of the liquid displaced by the body is 3 lbs. The same bulk of water would weigh 6 lbs.

$$\therefore s = \frac{4}{6} = \frac{2}{3}.$$

3. $s = \frac{\frac{1}{2}}{500} = \frac{1}{1000}$.

4. Call the weight of a cubic inch of water 1.

Then the copper weighs 36.
" zinc " 42.

$\therefore$ Each cubic inch of mixture weighs 7·8.

$$\therefore s = 7{\cdot}8.$$

§ 89.

1. The portion immersed is $1 - \frac{3}{40} = \frac{37}{40}$. $\therefore s = \frac{37}{40}$.

2. The same as that of water = 1.

3. The water displaced by $\frac{2}{3}$ of the body weighs 6 lbs.

∴ The water displaced by the whole body weighs 9 lbs. A force of 3 lbs. downwards must therefore be applied to counterbalance the upward pressure of the water.

§ 90.

1. By Boyle's Law the density is proportional to the pressure.
But when the pressure is 15 lbs., 570 measures the density.
∴ „ „ 1 lb., 38 „ „
∴ „ „ 16 lbs., 608 „ „
Or a cubic foot of air weighs 608 grains.
2. When the pressure is 1 the density is ·001.
∴ „ „ 1000 „ 1.
At 100 feet there is an *additional* pressure of 3 atmospheres.
∴ At 33,300 feet there is an additional pressure of 999 atmospheres.

Or the whole pressure at 33,300 feet is 1000 atmospheres. (This result does not represent the real state of the case, as Boyle's law is only approximately true, and the discrepancy increases with the pressure.)

§ 97.

The areas of the pistons are proportional to the squares of their diameters, or are as 9 to 2500.

If, then, $P = 72$, $W = \frac{2500 \times 72}{9} = 20{,}000$ lbs.

General Examples in Hydrostatics.

1. Iron is 29 times as dense as cork.

2. (1) 15·4 lbs. (2) 51·9 lbs. (3) 6890 lbs.

3. (1) 2230·3 lbs. (2) 72225 lbs. (3) 4,457,160 lbs.

4. As 6 to 5.

5. (1) $1687\frac{1}{2}$ lbs. (2) $21{,}127\frac{1}{2}$ lbs. (3) Yes, for the additional pressure is the same outside as inside.

6. (1) $16\frac{1}{2}$ lbs. (2) 2592 lbs.

7. (1) 15·2 lbs. (2) 16·3 lbs. (3) 28·1 lbs.

8. (1) $19\frac{1}{3}$. (2) $3\frac{1}{2}$. (3) $\frac{18}{25}$ or ·72.

9. 1 lb. 4 oz.

10. (1) $11\frac{1}{3}$. (2) $8\frac{1}{3}$. (3) 9. (4) 1·86.

11. (1) ·86. (2) ·66. (3) ·92.

12. (1) 54 cubic inches. (2) 53 cubic inches.

13. 32 lbs.

14. (1) 1·85. (2) 1·23. (3) ·81.

15. $6\frac{1}{4}$.

16. 39 cubic centimetres.

17. A, $\frac{1}{2}$, B, $1\frac{1}{4}$.

18. (1) $22\frac{1}{2}$. (2) $7\frac{1}{2}$. (3) $14\frac{1}{3}$. (4) $8\frac{2}{3}$.

19. $1\frac{1}{4}$.

20. $\dfrac{29}{120}$.

21. 800.

22. 6·912.

23. 2·6.

24. 3 cwt.

25. (1) 5. (2) $\frac{4}{5}$.

26. (1) $2\frac{1}{2}$ lbs. (2) $13\frac{1}{2}$ lbs. (3) $15\frac{1}{2}$ lbs.

27. 33·3 feet.

28. $11\frac{1}{3}$.

29. $1\frac{3}{13}$.

30. $\frac{4}{5}$.

31. $\frac{3}{13}$.

32. 23·68 cubic inches.

33. 14 lbs. 6·1 oz.
34. As $\frac{9}{5}$ to $\frac{10}{6}$, or 27 to 25.
35. 10·53.
36. 10,250 cubic feet.
37. 41 inches.
38. $\frac{1}{2}$ a cubic inch.
39. 4 tons.
40. (1) 2·92 cubic feet. (2) 75 ounces.
41. 271·32 inches.
42. It will be four times as great.
43. 8750.

UNIVERSITY EXAMINATION PAPERS.

I.

1. § 26.
2. §§ 49, 50.
3. (1) Prove as in § 70 that if AB represent the given weight, BC represents the force, and AC the pressure on the plane.
   (2) Compare Chapter V., General Example 36, where in a similar case it is proved that the bases are proportional to the weights. Therefore the base of the second plane is to be made twice that of the first.
4. § 57. Compare also Example 1.
5. § 68.
6. (1) See § 52 and § 53 (1).
   (2) By § 55 the pencil will fall as soon as the centre of gravity is beyond the base. The centre of gravity of the pencil is $\frac{1}{12}$ of its length within the table. The beetle may therefore crawl $\frac{4}{12}$ of the pencil's length beyond the table, when it will be $\frac{1}{12}$ from the projecting end.
7. (1) § 61.
   (2) § 62.
8. Part of § 75 and § 76.
9. § 82.

10. The cylinder contains air which has been compressed into half its volume, and therefore, by Boyle's law, the pressure has been doubled. The additional pressure is therefore that of one atmosphere, and the depth is the same as the height of the column of water in the water barometer (32 to 34 feet).

11. (1) Part of § 86 and § 87. See figure of § 85.
    (2) Since spheres are proportional to the cubes of their radii, the volume of the whole ball is eight times

that of the cork, or we have 7 volumes of gutta-percha to 1 of cork. Call the weight of 1 volume of water unity, then 1 volume of cork weighs ·24.

∴ 7 volumes of gutta percha weigh $7 \times \cdot 98 = 6 \cdot 86$.

∴ The 8 volumes weigh 7·1.

And 8 volumes of water weigh 8.

∴ The specific gravity of the whole is $\frac{71}{80}$, and $\frac{9}{80}$ of the whole floats above the surface.

Compare § 88, Ex. 5, and Ex. for Exercise 4. Also see § 89.

12. (1) § 91.
    (2) By the height of the column of mercury is meant the perpendicular distance between the level of the mercury in the tube and the level in the cistern. If this is measured in each case it is of no importance whether the tube is vertical. But if the height be read from a scale attached to the tube it is evident that this scale will only give correct results when the tube is vertical.

II.

1. (1) § 4.
   (2) While A runs $2x$ yards, B runs $3x$. If at this point B overtake A, then $3x = 2x + a$.
   $\therefore x = a$, and $2x = 2a$.
2. § 41.
3. (1) § 46.
   (2) See Chapter III., General Example 34 (3).
4. Chapter III., General Example 32. Compare § 17, Ex. 1.
5. (1) § 57.
   (2) § 57, Example for Exercise 2.
6. Chapter III., General Example 33.
7. (1) §§ 30, 34, 42.
   (2) $1 - m^2$.
8. (1) § 86; § 88, Example 5.
   (2) Let the weight of a volume of water be 1. Then the weight of one volume of the first fluid is 1, and of the second fluid 1·7. But the 2 volumes when mixed

occupy only $\frac{9}{10}$ of two volumes. Therefore the same volume of water weighs 1·8.

$$\therefore s = \frac{1 + 1\cdot7}{1\cdot8} = 1\tfrac{1}{2}.$$

9. §§ 92, 96.

10. (1) § 97.
 (2) 27 tons. See Chapter VI., General Example 39.

III.

1. (1) § 4.
 (2) 55. Compare § 5, Example 2.

2. $v = 80\cdot5$, $s = 100\frac{5}{8}$. Compare § 38, Ex. 1.

3. (1) § 25.
 (2) As 1 to 24. Compare § 25, Ex. 1.

4. §§ 34, 38.

5. The forces are equivalent to 4 northwards and 3 eastwards. The resultant is 5 in the direction of the diagonal of the parallelogram. Compare § 15, Ex.

6. At C, if AC = $\frac{3}{5}$ of AB. Compare § 49, Ex. 1.

7. § 56.

8. 13 lbs. See Chapter V., Gen. Ex. 31.

9. (1) § 75.
 (2) § 81.

10. The water supports $\frac{1}{11}$ of the weight.

$$\therefore \tfrac{10}{11} \text{ of the weight} = 1 \text{ lb.}$$

$$\therefore \text{The whole weight} = \tfrac{11}{10} \text{ lbs.} = 1\tfrac{1}{10} \text{ lbs.}$$

IV.

1. (1) § 41.
 (2) In figure of § 41, the resultant of AB and BC is AC, that of AC and CD is AD, that of AD and DA is 0. Similarly for a rectangle.

2. (1) By the second law of motion, forces acting on the same particle are represented by the velocities they produce; hence § 10 and § 18 may be applied to forces, or one force may be considered as equivalent to other two which produce the same result.

(2) Compare § 17, last paragraph. Total force = 2·7, vertical component = $\sqrt{(2{\cdot}7)^2 - (1{\cdot}35)^2}$ = 2·34 approx.

3. See § 62. Here $Wa = 9b$, $4a = Wb$.
$\therefore 4Wa^2 = 9Wb^2$. $\therefore 4a^2 = 9b^2$. $\therefore 2a = 3b$.

$$\text{Or } \frac{a}{b} = \frac{3}{2}.$$

4. The feather will be carried along at the same rate as the balloon, and will therefore appear to the man to fall vertically downwards.

5. The earth rotates from west to east. The stone though falling retains its eastward motion, which is quicker than that of those parts of the earth which are nearer the centre. Hence it strikes the east side.

6. See § 88, Ex. 1. Here $s = \dfrac{80-46}{80-56} = \dfrac{34}{24} = 1\tfrac{5}{12}$.

7. §§ 84, 85.

8. The average depth of the air in the cylinder is 15½ feet. The pressure is therefore that due to 34 + 15½ = 49½ feet of water. At this pressure the air occupies 15 cubic feet. At a pressure due to one foot, the volume would be 49½ × 15 ∴ At a pressure due to 34 feet it would be $\dfrac{49\frac{1}{2} \times 15}{34} = 21\tfrac{57}{68}$ cubic feet.

State Boyle's law, and the results of § 81.

V.

1. (1) § 26. (2) 50 lbs.

2. § 59.

3. (1) §§ 4, 21.
   (2) In equation (10) $V = 0$, $\therefore s = \frac{1}{2}at^2$.
   If $t = 1$, $s = \frac{1}{2}a = 10$, $\therefore a = 20$.
   If $t = 2$, $s = 2a = 40$.

4. § 66.

5. See § 43. Here $v = \dfrac{5280}{5} = 1056$.

$$\therefore V = \frac{5 \times 1056}{200} = 26{\cdot}4 \text{ feet per second.}$$

VI.

1. (1) § 26.
   (2) The forces of 3 and 7 are in the same direction, that of 10 in the opposite direction.
2. (1) § 41.
   (2) The triangle of forces is half an equilateral triangle.
   $\therefore P = 2R$, or $R = 50$, $Q = \sqrt{10{,}000 - 2500} = 86{\cdot}6$.

3. Let ABC be the rod in its position of equilibrium, B being its middle point. From A and B draw vertical lines AD, BE, meeting a horizontal line through C at D and E; join AE. Three forces act on the rod, its weight at B, the horizontal force at C, and the reaction at A. These forces must pass through one point (§ 49, Ex. for Exercise 3), and as the directions of the first two meet in E, AE is the direction of the reaction. These 3 forces are therefore in the directions of the 3 sides of the triangle ADE. Therefore DE = $\frac{1}{2}$ of AD. But DE is $\frac{1}{2}$ of DC. $\therefore$ DC = AD, or the rod is at an angle of 45° to the horizon.

4. (1) § 49.
   (2) Replace each weight by two weights of half the amount at the extremities of the sides. Thus the pressures on the legs are increased by $3 + 4 = 7$ lbs., $4 + 5 = 9$ lbs., and $5 + 3 = 8$ lbs.
5. (1) § 52 and § 53 (3).
   (2) Call the square ABCD. The weights at A and D are equivalent to 5 lbs. at a point E, $\frac{4}{5}$ along AD. The weights at B or C are equivalent to 5 lbs. at a point F, $\frac{3}{5}$ along BC. Therefore the centre of gravity of the whole is the middle point of EF.
7. (1) § 64.
   (2) 3 inches.

7. In the first case the slope of the plane is diminished, and therefore the force exerted is less (see § 70). In the second case, the centre of gravity of horse and man is kept above the base.

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TURNBULL AND SPEARS, PRINTERS, EDINBURGH.